

Computation and modeling in piecewise Chebyshevian spline spaces

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Abstract

A piecewise Chebyshevian spline space is *good for design* when it possesses a B-spline basis and this property is preserved under knot insertion. For such spaces, we construct a set of functions, called transition functions, which allow for efficient computation of the B-spline basis, even in the case of nonuniform and multiple knots. Moreover, we show how the spline coefficients of the representations associated with a refined knot partition and with a raised order can conveniently be expressed by means of transition functions. To illustrate the proposed computational approach, we provide several examples of interest in various applications, ranging from Geometric Modeling to Isogeometric Analysis.

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1. Introduction

Extended Chebyshev (EC) spaces represent a natural generalization of polynomials. They contain transcendental functions (and thus are capable of reproducing circles, ellipses, etc.) and provide additional degrees of freedom which can be exploited to control the behavior of parametric curves and to accomplish shape-preserving approximations. While Chebyshevian splines are piecewise functions whose segments belong to the same EC-space, functions having pieces in different EC-spaces are called *piecewise Chebyshevian splines*. The latter are of great interest for their capacity to combine the local nature of splines with the diversity of shape effects provided by the wide range of known EC-spaces. Their applications spans many disciplines, ranging from the classical domains of Geometric Design and Approximation Theory, up to Multiresolution Analysis [1–3] and Isogeometric Analysis [4, 5], where they have been more recently introduced.

Piecewise Chebyshevian splines can either be parametrically or geometrically continuous. The latter are

piecewise functions where the continuity conditions between adjacent segments are expressed by means of connection matrices [6]. In both cases, to be of interest for Geometric Design or other applications, a piecewise Chebyshevian spline space should possess a B-spline basis – in the usual sense of a normalised basis composed of minimally supported splines – and this feature should be maintained after knot insertion. The above properties are summarized by saying that a space is *good for design* and, as proved in [7] (see also [8, 9]), they are equivalent to existence of blossoms in the space. As a consequence of blossoms, the B-spline basis is the optimal normalized totally positive basis and all classical geometric design algorithms can be developed in the space.

In this paper we address the problem of efficiently computing and numerically evaluating the B-spline basis of a good for design spline space.

The reader should though be aware that determining whether or not a spline space is good for design is a separate (and sometimes difficult) problem, which is out of the scope of this paper. To this purpose there exist theoretical results (see [7] and references therein) yielding necessary and sufficient conditions. These can readily be used for spline spaces having sections of relatively low dimension (up to dimension four), but the computations become overly complicated for spaces of higher

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dimension. Regardless of the dimension and the kind of the underlying section spaces, the numerical approach in [10] provides sufficient conditions for a spline space to be (numerically) good for design.

General approaches for the computation of B-spline-type bases include the well-known recurrence relation for polynomial splines [11] and the formulae for the evaluation of trigonometric and hyperbolic splines proposed respectively in [12] and [13]. A generalized version of the polynomial recurrence relation, which applies to Chebyshevian splines, was given in [14] and reads as ¹

$$B_i^{1,m}(x) = \begin{cases} u(x), & \text{if } t_i \leq x < t_{i+1} \\ 0, & \text{otherwise} \end{cases},$$

$$B_i^{k+1,m} = \lambda_i^{k,m} B_i^{k,m} + (1 - \lambda_{i+1}^{k,m}) B_{i+1}^{k,m},$$

with

$$\lambda_i^{k,m} = (d_{i+1}^{k-1,m} - d_i^{k,m}) / (d_{i+1}^{k,m} - d_i^{k,m}),$$

where $B_i^{k+1,m}$ is the i -th basis function of order $k+1 \leq m$ and $d_i^{k,m}$ is a properly defined *Chebyshevian divided difference* of order k . It shall be noted that the computation of Chebyshevian divided differences requires to know a set of weight functions $\mathbf{w} = \{w_0, \dots, w_n\}$ for the canonical Extended Complete Tchebycheff (ECT) system which spans each local ECT-space.

An integral recurrence relation for defining Chebyshevian B-spline functions was developed in [15]. More precisely, given a knot sequence $\mathbf{t} = \{t_i\}$ and a sequence of integral-positive functions $\mathbf{w} = \{w_0, \dots, w_n\}$, the spline basis $A_i^n(x)$ of order $n+1$ over \mathbf{t} with respect to \mathbf{w} is recursively defined by

$$A_i^0(x) = \begin{cases} w_0(x), & \text{if } t_i \leq x < t_{i+1} \\ 0, & \text{otherwise} \end{cases},$$

$$A_i^n(x) = w_n(x) \int_{-\infty}^x (A_i^{n-1}(y) / \alpha_i^{n-1} - A_{i+1}^{n-1}(y) / \alpha_{i+1}^{n-1}) dy,$$

where $\alpha_j^{n-1} = \int_{-\infty}^{\infty} A_j^{n-1}(y) dy$. Similar integral relations were used to compute so-called GB-splines and Unified Extended splines in [16] and [17] respectively.

Whereas integral recurrence relations require the knowledge of a proper set of weight functions, if these functions are not given it is still possible to construct a B-spline basis by Hermite interpolation [18]. The method developed in this paper, which is based on the use of *transition functions*, follows along the same line

and does not require a priori knowledge of the weight functions. In particular, as we will see, by means of transition functions it is possible to construct the Bernstein basis of any EC-space or the B-spline basis of any piecewise Chebyshevian spline space good for design in a conceptually easier, computationally more efficient and more general way compared to alternative approaches.

For simplicity of presentation we work in the framework of parametric continuity and in the last part of the paper we briefly illustrate how the proposed results can naturally be extended to geometrically continuous splines.

It is interesting to remark that the idea of transition functions, in its first form, already appeared in the original work by Pierre Bézier [19]. Moreover, recently, a suitable generalization of this notion was used for the construction of local blending functions in the context of polynomial spline interpolation [20, 21].

The remainder of the paper is organized as follows. In Section 2 we introduce the transition functions, we discuss their relationship to the B-spline basis and illustrate how they can effectively be computed. In Sections 3 and 4 we exploit these functions to develop the classical modeling tools of knot-insertion and order-elevation for piecewise Chebyshevian splines. Section 5 lists the computational algorithms that can be drawn for the preceding discussion. Section 6 is devoted to illustrating some application examples and finally Section 7 discusses how the transition functions can be used in more general settings, including geometrically continuous splines, splines with knots of zero multiplicity, multi order splines and the recently studied (piecewise) quasi Chebyshevian splines [22].

2. Transition functions and B-spline bases for piecewise Chebyshevian spline spaces

In this section we introduce the definition of transition functions and exploit such functions to efficiently compute the B-spline basis of any piecewise Chebyshevian spline space good for design, regardless of the kind and dimension of the different section spaces.

2.1. Preliminary notions on EC-spaces

Let us proceed by recalling the definition of EC-spaces, which are the building blocks of the splines we are interested in.

Definition 1 (Extended Chebyshev space). An m -dimensional space \mathcal{U} contained in $C^{m-1}(I)$, $I \subset \mathbb{R}$, is an *Extended Chebyshev space* (for short, EC-space) on I if

¹The formulas in this section are presented in the original notation.

any nonzero element of \mathcal{U} vanishes at most $m - 1$ times in I , counting multiplicities as far as possible for C^{m-1} functions (that is, up to m), or, equivalently, if any Hermite interpolation problem in m data in I has a unique solution in \mathcal{U} .

Hereinafter we refer to the basis functions spanning an EC-space as its *EC-system*. Subdividing the domain interval into a number of consecutive subintervals, it is possible to introduce the definitions of piecewise Chebyshevian splines and Chebyshevian splines. A piecewise Chebyshevian spline is one whose sections belong to different EC-spaces. In contrast, a Chebyshevian spline is characterized by having all sections in the same EC-space.

In this paper we will mainly consider piecewise Chebyshevian splines where adjacent spline pieces are connected via the standard parametric continuity. Moreover, we focus our attention on such spaces that are good for design [7]. Indeed, this characterization, not only restricts us to the class of spaces that may have a practical interest in applications, but also provides us with all the necessary properties to be used in the next sections. As previously recalled, a piecewise Chebyshevian spline space is good for design when it possesses blossom. An equivalent definition is the following.

Definition 2. A piecewise Chebyshevian spline space is *good for design* if it possesses the B-spline basis and so does any spline space obtained from it by knot insertion.

2.2. Construction of the B-spline basis

This section presents a general and efficient procedure to construct the B-spline basis for any piecewise Chebyshevian spline space good for design. This property also guarantees that we can use such spline space for spline interpolation.

In the following we use the terms *break-point*, *knot* and *node* in the following sense: a break-point is the junction between two spline segments, a knot corresponds to a break-point repeated as many times as its multiplicity and a node is the abscissa of an interpolation point.

Let $[a, b]$ be a bounded and closed interval, and $\Delta := \{x_i\}_{i=1, \dots, q}$ be a set of *break-points* such that $a \equiv x_0 < x_1 < \dots < x_q < x_{q+1} \equiv b$. Let us consider the partition of $[a, b]$ induced by the set Δ into the subintervals $[x_i, x_{i+1}]$, $i = 0, \dots, q$. Moreover, let m be a positive integer, and $\mathcal{M} := (\mu_1, \dots, \mu_q)$ be a vector of positive integers such that $1 \leq \mu_i < m$ for every $i = 1, \dots, q$. We denote by $\mathcal{U}_m := \{\mathcal{U}_{0,m}, \dots, \mathcal{U}_{q,m}\}$ an ordered set

of spaces of dimension m such that every $\mathcal{U}_{i,m}$ is an EC-space on the interval $[x_i, x_{i+1}]$ containing constants. Furthermore, we require that the space $D\mathcal{U}_{i,m} := \{Du := u' \mid u \in \mathcal{U}_{i,m}\}$ be an $(m - 1)$ -dimensional EC-space on $[x_i, x_{i+1}]$, $i = 0, \dots, q$.

Definition 3 (Piecewise Chebyshevian splines). We define the set of *piecewise Chebyshevian splines* of order m with break-points x_1, \dots, x_q of multiplicities μ_1, \dots, μ_q as

$$S(\mathcal{U}_m, \mathcal{M}, \Delta) := \{s \mid \text{there exist } s_i \in \mathcal{U}_{i,m}, i = 0, \dots, q, \text{ such that:}$$

$$\text{i) } s(x) = s_i(x) \text{ for } x \in [x_i, x_{i+1}], i = 0, \dots, q;$$

$$\text{ii) } D^\ell s_{i-1}(x_i) = D^\ell s_i(x_i) \text{ for } \ell = 0, \dots, m - \mu_i - 1, \\ i = 1, \dots, q \}.$$

It follows from standard arguments that $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ defined above is a function space of dimension $m + K$ with $K := \sum_{i=1}^q \mu_i$ [23].

Definition 4 (Extended partition). The set of knots $\Delta^* := \{t_i\}_{i=1, \dots, 2m+K}$, with $K = \sum_{i=1}^q \mu_i$, is called an *extended partition* associated with $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ if and only if:

- i) $t_1 \leq t_2 \leq \dots \leq t_{2m+K}$;
- ii) $t_m \equiv a$ and $t_{m+K+1} \equiv b$;
- iii) $\{t_{m+1}, \dots, t_{m+K}\} \equiv \underbrace{\{x_1, \dots, x_1\}}_{\mu_1 \text{ times}}, \dots, \underbrace{\{x_q, \dots, x_q\}}_{\mu_q \text{ times}}.$

For simplicity and without loss of generality, we will confine our discussion to the case of extended partitions without external additional knots, i.e. assuming $x_0 \equiv a$, $x_{q+1} \equiv b$ and $\mu_0 = \mu_{q+1} = m$. Piecewise Chebyshevian splines can similarly be constructed from a general partition with external knots and break-points. In the latter case, we shall assign an auxiliary EC-space to each interval which is not contained in $[a, b]$, in such a way that the continuity conditions between adjacent spline pieces are well-defined.

For a piecewise Chebyshevian spline space, the B-spline basis is defined as follows (see [7]).

Definition 5 (B-spline basis). A sequence $\{N_{i,m}\}_{i=1, \dots, m+K}$ of elements of $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ is the *B-spline basis* if it meets the following requirements:

- i) *support property*: for each $i \in \{1, \dots, m + K\}$, $N_{i,m}(x) = 0$ for $x \notin [t_i, t_{i+m}]$;

- ii) *positivity property*: for each $i \in \{1, \dots, m+K\}$, $N_{i,m}(x) > 0$ for $x \in (t_i, t_{i+m})$;
- iii) *endpoint property*: for each $i \in \{1, \dots, m+K\}$, $N_{i,m}$ vanishes exactly $m - \mu_i^R$ times at t_i and exactly $m - \mu_{i+m}^L$ times at t_{i+m} where

$$\begin{aligned}\mu_i^R &:= \max\{j \geq 0 \mid t_i = t_{i+j}\} + 1 \quad \text{and} \\ \mu_i^L &:= \max\{j \geq 0 \mid t_{i-j} = t_i\} + 1;\end{aligned}$$

- iv) *normalization property*:

$$\sum_i N_{i,m}(x) = 1, \quad \forall x \in [a, b].$$

In addition, we will refer to a basis that has the above properties i) and iii) as a B-spline-like basis or a positive B-spline like basis if also ii) holds.

As usual, any spline $s \in S(\mathcal{U}_m, \mathcal{M}, \Delta)$ can be expressed as a linear combination of the B-spline basis functions $N_{i,m}$, $i = 1, \dots, m+K$, of the form

$$s(x) = \sum_{i=1}^{m+K} c_i N_{i,m}(x), \quad x \in [a, b],$$

or also, locally, as

$$s(x) = \sum_{i=\ell-m+1}^{\ell} c_i N_{i,m}(x), \quad x \in [t_\ell, t_{\ell+1}). \quad (1)$$

In the remainder of the section we introduce the transition functions for the space $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ and discuss the relationship between these functions and the B-spline basis. In particular we will see that they provide an efficient tool for computation of the B-spline basis.

Definition 6 (Transition functions). Let $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ be a Piecewise Chebyshevian spline space of dimension $m+K$ with $\Delta = \{x_i\}_{i=1, \dots, q}$ a partition of $[a, b]$ and $\Delta^* = \{t_i\}_{i=1, \dots, 2m+K}$ the associated extended partition. Let also suppose that $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ has a B-spline basis $\{N_{i,m}\}_{i=1, \dots, m+K}$. We call *transition functions* the piecewise functions f_i , given by:

$$f_i = \sum_{j=i}^{m+K} N_{j,m}, \quad i = 1, \dots, m+K. \quad (2)$$

It is immediate to verify that the transition functions are a basis for $S(\mathcal{U}_m, \mathcal{M}, \Delta)$. One way, is to write (2) as

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{m+K} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} N_{1,m} \\ N_{2,m} \\ \vdots \\ N_{m+K,m} \end{pmatrix} \quad (3)$$

and observe that the above matrix is nonsingular. Moreover, assuming $f_{m+K+1} \equiv 0$, we can reverse relation (3) expressing the B-spline basis in terms of transition functions as

$$N_{i,m} = f_i - f_{i+1}, \quad i = 1, \dots, m+K. \quad (4)$$

From the properties of the B-spline basis listed in Definition 5 we can deduce that $f_1(x) = 1$, for all $x \in [a, b]$, and that the piecewise functions f_i , $i = 2, \dots, m+K$, have the following characteristics:

a)

$$f_i(x) = \begin{cases} 0, & x \leq t_i, \\ 1, & x \geq t_{i+m-1}, \end{cases}$$

- b) if $t_i < t_{i+m-1}$, f_i vanishes $m - \mu_i^R$ times at t_i and $1 - f_i$ vanishes $m - \mu_{i+m-1}^L$ times at t_{i+m-1} .

In addition, the following bounds on the number of zero derivatives hold

$$D_+^r f_i(t_i) = 0, \quad r = 0, \dots, k_i^R,$$

$$D_+^{k_i^R+1} f_i(t_i) > 0,$$

and

$$D_-^r f_i(t_{i+m-1}) = \delta_{r,0}, \quad r = 0, \dots, k_{i+m-1}^L,$$

$$(-1)^{k_{i+m-1}^L} D_-^{k_{i+m-1}^L+1} f_i(t_{i+m-1}) > 0,$$

where

$$k_i^R := m - \mu_i^R - 1 \quad \text{and} \quad k_{i+m-1}^L := m - \mu_{i+m-1}^L - 1. \quad (5)$$

The above properties a) and b) provide a practical way of computing the transition functions provided that the spline space $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ is good for design. As previously recalled, the latter assumption guarantees that we can use such spline space for spline interpolation. Moreover, this is also true in the restriction of $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ to any interval contained in $[a, b]$. From a) it is immediate to see that each transition function f_i , $i = 2, \dots, m+K$, has a nontrivial expression (i.e. it is neither the constant function zero or one) in $[t_i, t_{i+m-1}]$. Hence, the restriction of f_i to such interval can be determined by Hermite interpolation using b).

More precisely, let p_i be the index of the break-point associated to the knot t_i and, supposed $t_i < t_{i+m-1}$, let us denote by $x_{p_i}, \dots, x_{p_{i+m-1}}$ the break-points of Δ contained in $[t_i, t_{i+m-1}]$. Then the function f_i consists in $p_{i+m-1} -$

p_i pieces and shall satisfy the continuity conditions and endpoint properties

$$\begin{aligned} D^r f_{i,p_i}(x_{p_i}) &= 0, & r &= 0, \dots, k_i^R, \\ D^r f_{i,j-1}(x_j) &= D^r f_{i,j}(x_j), & r &= 0, \dots, k_j, \\ & & j &= p_i + 1, \dots, p_{i+m-1} - 1, \\ D^r f_{i,p_{i+m-1}-1}(x_{p_{i+m-1}}) &= \delta_{r,0}, & r &= 0, \dots, k_{i+m-1}^L, \end{aligned} \quad (6)$$

where $f_{i,j}$, $j = p_i, \dots, p_{i+m-1} - 1$, is the restriction of f_i to the interval $[x_j, x_{j+1}]$, k_i^R and k_{i+m-1}^L are given in (5) and $k_j := m - \mu_j - 1$.

The above equations (6) uniquely determine f_i . To justify this assertion, we shall observe that the number of conditions in (6) is equal to the dimension of the restriction of $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ to $[t_i, t_{i+m-1}]$ and that the nodes and knots satisfy proper interlacement conditions [18] - which are similar to those required for polynomial splines. In the considered setting, these two requirements are satisfied regardless of the kind and dimension of the section spaces and of the multiplicity of knots.

$$A := \begin{pmatrix} A_{p_i}(x_{p_i}) & & & & \\ A_{p_i}(x_{p_i+1}) & -A_{p_i+1}(x_{p_i+1}) & & & \\ & A_{p_i+1}(x_{p_i+2}) & -A_{p_i+2}(x_{p_i+2}) & & \\ & & \ddots & \ddots & \\ & & & A_{p_{i+m-1}-2}(x_{p_{i+m-1}-1}) & -A_{p_{i+m-1}-1}(x_{p_{i+m-1}-1}) \\ & & & & A_{p_{i+m-1}-1}(x_{p_{i+m-1}}) \end{pmatrix}$$

$$\begin{aligned} \mathbf{b} &:= (b_{i,1}, \dots, b_{i,m(p_{i+m-1}-p_i)})^T, \\ \mathbf{c} &:= (0, \dots, 0, 1, \underbrace{0, \dots, 0}_{k_{i+m-1}^L \text{ times}})^T, \end{aligned}$$

and where $A_j(x_h)$, $h = j, j+1$, is a matrix of dimensions $(k_h + 1) \times m$ whose r th row, $r = 1, \dots, k_h + 1$, is

$$(D^{r-1} u_{j,1}(x_h), \dots, D^{r-1} u_{j,m}(x_h)).$$

As a consequence, it shall be noted that, once chosen the generators of the local space $\mathcal{U}_{i,m}$, also their derivatives must be provided for the setup of the linear systems (8).

Remark 1 (Derivatives of a transition function). *It shall be noted that the solution of the above system immediately yields the derivatives of the transition functions up to any order. In fact, these can be expressed as a linear combination of the derivatives of the EC-systems with coefficients \mathbf{b} .*

In practice, to compute the i -th transition function f_i we shall proceed as follows. Suppose that, on $[x_j, x_{j+1}]$, $j = p_i, \dots, p_{i+m-1} - 1$, the spline space is locally spanned by an EC-system

$$\{u_{j,1}, u_{j,2}, \dots, u_{j,m}\},$$

and denote by $b_{i,s}$, $s = 1, \dots, m(p_{i+m-1} - p_i)$ the coefficients of the local expansion of $f_{i,j}$ in such system. In other words, we will have

$$\begin{aligned} f_{i,j}(x) &= \sum_{h=1}^m b_{i,r+h} u_{j,h}(x), & x &\in [x_j, x_{j+1}], \\ & & j &= p_i, \dots, p_{i+m-1} - 1, \\ & & r &= m(j - p_i). \end{aligned} \quad (7)$$

Hence, according to (6), the coefficients of such expansion can be determined by solving a linear system of the form

$$A\mathbf{b} = \mathbf{c}, \quad (8)$$

with

Remark 2 (Bernstein basis). *If the knot partition is empty (i.e., $\Delta = \emptyset$), the piecewise Chebyshevian spline space is an EC-space and formula (4) yields its Bernstein basis. In this case, we will indicate the order $n+1$ basis functions with $B_{0,n}, \dots, B_{n,n}$, as is usual when dealing with Bernstein bases. Consistently, and to distinguish them from their spline counterpart, we will label the transition functions g_0, \dots, g_n . In particular, assuming that $g_{n+1} \equiv 0$, relation (4) reads now as*

$$B_{i,n} = g_i - g_{i+1}, \quad i = 0, \dots, n. \quad (9)$$

Some remarks on how to choose the local EC-systems are in order. At the beginning of this section, we have assumed that every section $\mathcal{U}_{i,m}$, for all $i = 0, \dots, q$, contains constants and that both $\mathcal{U}_{i,m}$ and $D\mathcal{U}_{i,m}$ are EC-spaces on $[x_i, x_{i+1}]$. These requirements guarantee that $\mathcal{U}_{i,m}$ itself is good for design and admits a Bernstein basis. Then the natural choice for the local EC-system $u_{i,1}, \dots, u_{i,m}$ is indeed to take such a basis.

The Bernstein basis can be given explicitly, if known, or computed resorting to the transition functions. In the latter case, we can either obtain the basis functions by symbolic computation (for relatively simple spaces, e.g. in the case of low dimension) or proceed numerically.

Figure 1 shows a set of transition functions and the related B-spline basis functions. In addition, we discuss below two instructive examples, in which the proposed approach serves as an alternative construction for the well-known polynomial B-spline and Bernstein bases.

Example 1 (Polynomial B-splines). Let us consider the space of polynomial splines $S(\mathcal{P}_m, \mathcal{M}, \Delta)$, where each piece is spanned by the space \mathcal{P}_m of polynomials of order m (i.e., of degree at most $m - 1$). For the sake of simplicity, we will assume that each break-point in Δ has multiplicity equal to 1. Then the transition functions are the order- m splines such that $f_1 \equiv 1$ and f_i , $i = 2, \dots, m + K$, is the unique solution of the linear system

$$\begin{aligned} D^r f_{i,i}(t_i) &= 0, & r &= 0, \dots, m-2, \\ D^r f_{i,j-1}(t_j) &= D^r f_{i,j}(t_j), & r &= 0, \dots, m-2, \\ & & j &= i+1, \dots, i+m-2, \\ D^r f_{i,i+m-2}(t_{i+m-1}) &= \delta_{r,0}, & r &= 0, \dots, m-2. \end{aligned}$$

Once solved the above systems, the B-spline basis functions $N_{i,m}$ are given by (4).

Example 2 (Polynomial Bernstein basis). The Bernstein basis for the space \mathcal{P}_m of polynomials of order m on a given interval $[a, b]$ can be derived considering an extended partition of the form

$$\underbrace{\{a, \dots, a\}}_{m \text{ times}}, \underbrace{\{b, \dots, b\}}_{m \text{ times}}.$$

The corresponding set of transition functions consists of $g_0 \equiv 1$ and of the functions g_i , $i = 1, \dots, n$, determined by the following conditions:

$$\begin{aligned} D^r g_i(a) &= 0, & r &= 0, \dots, i-1, \\ D^r g_i(b) &= \delta_{r,0}, & r &= 0, \dots, n-i. \end{aligned}$$

According to (9), the Bernstein polynomials are computed as $B_{i,n} = g_i - g_{i+1}$, $i = 0, \dots, n-1$, $B_{n,n} = g_n$.

In the remainder of this section we will illustrate the advantage of using the transition functions for the computation of Bernstein or B-spline type bases in terms of number of operations to be performed. We compare the proposed approach with the other fully general method that does not require knowledge of a set of weight functions. The latter consists in first determining by Hermite interpolation a positive basis that satisfies

suitable endpoint conditions (i.e. a positive Bernstein-like or B-spline like basis) and then imposing the partition of unity property. We will refer to this procedure as the “classical” one and we will show that the proposed method entails less computations. This can be intuitively understood taking into account two basic facts: 1) the use of transition functions does not require an additional normalization step; 2) whereas a transition function f_i is nontrivial in the interval $[t_i, t_{i+m-1}]$, a B-spline-like basis function $N_{i,m}$ is nontrivial in $[t_i, t_{i+m}]$. Therefore, to work out one transition function less intervals (and conditions) are needed.

In the following we will compare the two approaches in greater detail. To this aim, it is useful to consider at first the simpler problem of computing the Bernstein basis of an $(n+1)$ -dimensional EC-space. According to the classical approach, a set of positive Bernstein-like basis functions $\bar{B}_{i,n}$, $i = 0, \dots, n$ can be determined by solving the $(n+1)$ -dimensional linear systems

$$\begin{aligned} D^r \bar{B}_{i,n}(a) &= 0, & r &= 0, \dots, i-1 \\ D^i \bar{B}_{i,n}(a) &= 1, \\ D^r \bar{B}_{i,n}(b) &= 0, & r &= 0, \dots, n-i-1, \end{aligned}$$

for $i = 0, \dots, n$. Hence, the Bernstein basis functions will be given by $B_{i,n}(x) = \alpha_i \bar{B}_{i,n}(x)$, for some positive α_i , $i = 0, \dots, n$, and therefore there should hold

$$\sum_{i=0}^n \alpha_i D^r \bar{B}_{i,n}(a) = \delta_{r,0}, \quad r = 0, \dots, n.$$

To determine the coefficients α_i from the above equations we need to solve another $(n+1)$ -dimensional linear system with the following lower triangular matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ D\bar{B}_{0,n}(a) & D\bar{B}_{1,n}(a) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D^n \bar{B}_{0,n}(a) & D^n \bar{B}_{1,n}(a) & \dots & D^n \bar{B}_{n,n}(a) \end{pmatrix}.$$

All together the procedure entails solving $n+2$ systems of dimension $n+1$ each. On the other hand, in terms of transition functions, we shall simply set $g_0 \equiv 1$ and determine g_i , $i = 1, \dots, n$ by solving n linear systems of size n only. After this step, the basis functions $B_{i,n}$, $i = 0, \dots, n$, are immediately given by relation (9).

An analogous reasoning can be repeated with a view to the computation of the B-spline basis of a spline space. In the classical setting, we shall first determine a positive B-spline-like basis $\{\bar{N}_{i,m}\}_{i=1, \dots, m+K}$ satisfying i), ii), iii) in Definition 5. The calculation of each basis function $\bar{N}_{i,m}$, $i = 1, \dots, m+K$, requires solving a linear

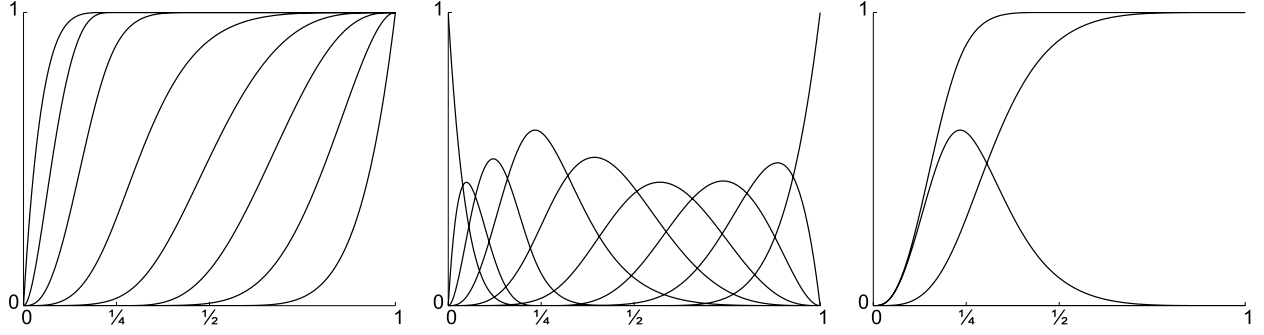


Figure 1: Transition functions (left) and B-spline basis (center) for the spline space $S(\mathcal{U}_6, \mathcal{M}, \Delta)$ with extended partition $\Delta^* = \{0, 0, 0, 0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 1, 1, 1, 1, 1, 1\}$ and $\mathcal{U}_{i,6} = \text{span}\{1, t, \cos t, \sin t, \cosh t, \sinh t\}$ with $t = x - x_i$ and $x \in [x_i, x_{i+1}]$ for all i . The figure on the right shows a single B-spline basis function and the two transition functions involved in its construction according to (4).

system of dimension $m(p_{i+m} - p_i)$ (for instance, in the case of simple knots we need to solve $m + K$ linear systems of dimension m^2). The successive normalization stage involves a lower triangular system of dimension $m + K$ having the form

$$\sum_{i=1}^{r+1} \alpha_i D^r \tilde{N}_{i,m}(a) = \delta_{r,0}, \quad r = 0, \dots, m-1,$$

$$\sum_{j=i-m+1}^i \alpha_j D^{m-\mu_i^R} \tilde{N}_{j,m}(t_i) = 0, \quad i = m+1, \dots, m+K.$$

On the other hand, using the transition functions and equation (4), we always have $f_1 \equiv 1$ and thus we are left with $m+K-1$ systems to be solved for the remaining f_i , $i = 2, \dots, m+K$. The computation of f_i entails solving a system of size $m(p_{i+m-1} - p_i)$ (for instance, when all knots are simple, $m+K-1$ linear systems of dimension $m(m-1)$ have to be solved). As a consequence, the classical approach involves the solution of a larger number of linear systems having higher dimensions.

To conclude this section we would like to point out another important computational advantage related to the use of transition functions. In some design algorithms, such as knot-insertion and order elevation, it is often required to compute a subset of the B-spline basis functions only. In this respect, even if we were to determine a single basis function $N_{j,m}$, the classical method would require to compute the entire B-spline-like basis beforehand. This is due to the additional normalization step, which involves all the elements of such basis. Conversely, computing $N_{j,m}$ from equation (4) only requires the two transition functions f_j and f_{j+1} , each of which can be constructed in an independent way.

3. Knot insertion

In this section we will see how to use the transition functions to perform knot insertion in a piecewise

Chebyshevian spline space good for design and illustrate the benefits deriving from this approach.

Our objective is thus to express the B-spline basis functions as nonnegative finite linear combinations of an analogous basis defined on a finer partition. For this to happen, we shall suppose that the space on the finer partition contains the space on the coarser one. We prefer not to dwell on this aspect now, to avoid interrupting the flow of presentation. We will devote the subsequent Section 3.1 to discussing how to suitably choose the spaces in order to make sure that this inclusion is satisfied.

Proposition 1. *Let $\Delta^* = \{t_i\}_{i=1, \dots, 2m+K}$ be an extended partition. By inserting a new knot \hat{t} in Δ^* , $t_\ell \leq \hat{t} < t_{\ell+1}$, we obtain a new knot partition $\hat{\Delta}^* = \{\hat{t}_i\}_{i=1, \dots, 2m+K+1}$. If the associated spline spaces are such that $S(\mathcal{U}_m, \mathcal{M}, \Delta) \subset S(\mathcal{U}_m, \mathcal{M}, \hat{\Delta})$, then, denoting by $N_{j,m}$ and $\hat{N}_{j,m}$ the B-spline basis functions on the knot partitions Δ^* and $\hat{\Delta}^*$ respectively, there exist coefficients $\alpha_{1,m}, \dots, \alpha_{m+K+1,m}$ with $0 \leq \alpha_{i,m} \leq 1$, $i = 1, \dots, m+K+1$, such that*

$$N_{i,m}(x) = \alpha_{i,m} \hat{N}_{i,m}(x) + (1 - \alpha_{i+1,m}) \hat{N}_{i+1,m}(x), \quad i = 1, \dots, m+K, \quad x \in \mathbb{R}. \quad (10)$$

In particular,

$$\alpha_{i,m} = \begin{cases} 1, & i \leq \ell - m + 1, \\ \frac{D_+^{k_i^R+1} f_{i,p_i}(t_i)}{D_+^{k_i^R+1} \hat{f}_{i,p_i}(t_i)}, & \ell - m + 2 \leq i \leq \ell - r + 1, \\ 0, & i \geq \ell - r + 2, \end{cases} \quad (11)$$

where f_{i,p_i} and \hat{f}_{i,p_i} are the first non-trivial pieces of the transition functions that define respectively the B-spline functions $N_{i,m}$ and $\hat{N}_{i,m}$, k_i^R is given in (5), and $1 \leq r < m$ represents the multiplicity of \hat{t} in $\hat{\Delta}^*$.

Proof. By (10) we can write:

$$f_i - f_{i+1} = \alpha_{i,m}(\hat{f}_i - \hat{f}_{i+1}) + (1 - \alpha_{i+1,m})(\hat{f}_{i+1} - \hat{f}_{i+2}). \quad (12)$$

Hence the coefficients $\alpha_{i,m}$ can be obtained by differentiating $k_i^R + 1$ times the expression (12) and evaluating the result at t_i . \square

Remark 3. The knot-insertion coefficients $\alpha_{i,m}$ in (11) are determined as the ratio of derivatives of the transition functions, which requires that these derivatives be nonzero at the evaluation point t_i . Observing that, at t_i , both f_i and \hat{f}_i have the same order of continuity k_i^R , there follows that we shall take their derivatives of order $k_i^R + 1$. The right-hand side evaluation is thus explained by the fact that the considered functions are continuous up to order k_i^R only at t_i .

Remark 4. An equivalent way for computing the coefficients α_i in (11) is:

$$\alpha_{i,m} = \begin{cases} 1, & i \leq \ell - m + 1, \\ \frac{D_-^{k_{i+m-1}^L+1} f_{i,p_{i+m-1}-1}(t_{i+m-1})}{D_-^{k_{i+m-1}^L+1} \hat{f}_{i+1,p_{i+m-1}}(t_{i+m-1})}, & \begin{cases} \ell - m + 2 \leq i, \\ i \leq \ell - r + 1, \end{cases} \\ 0, & i \geq \ell - r + 2. \end{cases} \quad (13)$$

Formula (13) is useful when the additional knots in the extended partition are not coincident. In this case, we can conveniently use it for the insertion of a knot $\hat{t} \equiv b$ in order to switch from an extended partition with distinct additional knots to one with coincident additional knots. Moreover we can use formula (13) to insert a knot at a location $\hat{t} > b$, i.e. external to the interval $[a, b]$.

Corollary 1. Under the same assumptions of Proposition 1, let us consider a spline

$$s(x) = \sum_{i=1}^{m+K} c_i N_{i,m}(x) = \sum_{i=1}^{m+K+1} \hat{c}_i \hat{N}_{i,m}(x).$$

Then, from equation (10) it follows that

$$\hat{c}_i = \begin{cases} c_i, & i \leq \ell - m + 1, \\ \alpha_{i,m} c_i + (1 - \alpha_{i,m}) c_{i-1}, & \begin{cases} \ell - m + 2 \leq i, \\ i \leq \ell - r + 1, \end{cases} \\ c_{i-1}, & i \geq \ell - r + 2, \end{cases} \quad (14)$$

with $\alpha_{i,m}$, $i = \ell - m + 2, \dots, \ell - r + 1$, as in (11).

We would like to draw the reader's attention to the computational advantages resulting from the use of transition functions. After inserting a new knot, the spline space on the coarse partition and the one on the refined

partition will have most transition functions in common, whereas only a limited number of them will change. Since each transition function is independently computed, it will be sufficient to recalculate only the ones which change. In particular, formula (11) comprises the two following parts. One is the evaluation of the numerators, which amounts to computing the derivatives of the transition functions f_i . As discussed in Remark 1, this operation is straightforward. The other part concerns the computation of the denominators, for which we will only need to calculate the transition functions affected by the insertion of the knot \hat{t} .

3.1. How to perform knot insertion in piecewise Chebyshevian spline spaces

When inserting a knot \hat{t} in $[x_\ell, x_{\ell+1})$ we shall make sure that the generated space $S(\hat{\mathcal{U}}_m, \hat{\mathcal{M}}, \hat{\Delta})$ contains the initial space $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ in order to be able to exploit formulas (10)–(11). This entails that the section spaces $\hat{\mathcal{U}}_{i,m}$ must be properly chosen. Clearly, when $\hat{t} = x_\ell$ it is sufficient to set $\hat{\mathcal{U}}_{i,m} = \mathcal{U}_{i,m}$, for all i . When $x_\ell < \hat{t} < x_{\ell+1}$, we shall set $\hat{\mathcal{U}}_{i,m} = \mathcal{U}_{i,m}$ for any $i < \ell$ and $\hat{\mathcal{U}}_{i,m} = \mathcal{U}_{i-1,m}$ for any $i > \ell + 1$. The choice of the EC-systems that generate $\hat{\mathcal{U}}_{\ell,m}$ and $\hat{\mathcal{U}}_{\ell+1,m}$ requires more attention and this section is devoted to discussing how this can be accomplished.

We start by observing that the sought inclusion of $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ in $S(\hat{\mathcal{U}}_m, \hat{\mathcal{M}}, \hat{\Delta})$ is trivially guaranteed if we take $\hat{\mathcal{U}}_{\ell,m}$ and $\hat{\mathcal{U}}_{\ell+1,m}$ to be the restriction of $\mathcal{U}_{\ell,m}$ to the two subintervals $[x_\ell, \hat{t})$ and $[\hat{t}, x_{\ell+1})$. This observation, which may seem odd at first glance, is motivated by the usual association of the operation of knot insertion with refinable spaces. In this respect we shall recall that there exist piecewise Chebyshevian spline spaces that are not refinable in the classical sense, but where we can still express the B-spline basis of $S(\hat{\mathcal{U}}_m, \hat{\mathcal{M}}, \hat{\Delta})$ in terms of the same basis of $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ through formulas (10)–(11). An example of this latter situation is a spline space where $\mathcal{U}_{\ell,m} = \text{span}\{1, t, \frac{(1-t)^3}{1+(v_{1,\ell}-3)(1-t)t}, \frac{t^3}{1+(v_{2,\ell}-3)(1-t)t}\}$, with $v_{1,\ell}, v_{2,\ell} \geq 3$, $t = \frac{x-x_\ell}{x_{\ell+1}-x_\ell}$ and $x \in [x_\ell, x_{\ell+1})$ [24]. In this case the only way to guarantee that the EC-system generating $\mathcal{U}_{\ell,m}$ and $\hat{\mathcal{U}}_{\ell+1,m}$ on the two intervals $[x_\ell, \hat{t})$ and $[\hat{t}, x_{\ell+1})$ is to choose:

- $\hat{\mathcal{U}}_{\ell,m} = \text{span}\{1, t, \frac{(1-t)^3}{1+(v_{1,\ell}-3)(1-t)t}, \frac{t^3}{1+(v_{2,\ell}-3)(1-t)t}\}$, with $x \in [x_\ell, \hat{t})$, $t = \frac{x-x_\ell}{x_{\ell+1}-x_\ell}$,
- $\hat{\mathcal{U}}_{\ell+1,m} = \text{span}\{1, t, \frac{(1-t)^3}{1+(v_{1,\ell}-3)(1-t)t}, \frac{t^3}{1+(v_{2,\ell}-3)(1-t)t}\}$, with $x \in [\hat{t}, x_{\ell+1})$, $t = \frac{x-x_\ell}{x_{\ell+1}-x_\ell}$,

$\nu_{1,\ell}, \nu_{2,\ell}$ being the same as for $\mathcal{U}_{\ell,m}$.

For most piecewise Chebyshevian spline spaces, however, the above approach is not the only viable one. As an example, let us consider the case of trigonometric splines, where $\mathcal{U}_{\ell,m} = \text{span}\{1, t, \cos(\theta_\ell t), \sin(\theta_\ell t)\}$, with $\theta_\ell \in (0, \pi)$, $t = \frac{x-x_\ell}{x_{\ell+1}-x_\ell}$ and $x \in [x_\ell, x_{\ell+1})$. Undoubtedly we can proceed analogously as above. However, in this situation (as well as for all spaces of mixed algebraic/trigonometric/hyperbolic splines) it is more usual to assign a different EC-system to each interval $[x_\ell, \hat{t}]$ and $[\hat{t}, x_{\ell+1})$, in such a way that the space $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ is reproduced. In the considered example we will set

- $\hat{\mathcal{U}}_{\ell,m} = \text{span}\{1, t, \cos(\hat{\theta}_\ell t), \sin(\hat{\theta}_\ell t)\}$, with $\hat{\theta}_\ell = \theta_\ell \frac{\hat{t}-x_\ell}{x_{\ell+1}-x_\ell}$, $t = \frac{x-x_\ell}{\hat{t}-x_\ell}$ and $x \in [x_\ell, \hat{t})$,
- $\hat{\mathcal{U}}_{\ell+1,m} = \text{span}\{1, t, \cos(\hat{\theta}_{\ell+1} t), \sin(\hat{\theta}_{\ell+1} t)\}$, with $\hat{\theta}_{\ell+1} = \theta_\ell \frac{x_{\ell+1}-\hat{t}}{x_{\ell+1}-x_\ell}$, $t = \frac{x_{\ell+1}-x}{x_{\ell+1}-\hat{t}}$ and $x \in [\hat{t}, x_{\ell+1})$.

Note that $\hat{\theta}_\ell$ and $\hat{\theta}_{\ell+1}$ are proportional to the lengths of $[x_\ell, \hat{t}]$ and $[\hat{t}, x_{\ell+1}]$ respectively. It is immediate to see that this choice still guarantees that $\mathcal{U}_{\ell,m}$ can be represented in terms of $\hat{\mathcal{U}}_{\ell,m}$ and $\hat{\mathcal{U}}_{\ell+1,m}$ on $[x_\ell, \hat{t})$ and $[\hat{t}, x_{\ell+1})$ respectively. For spaces where we can adopt the latter strategy, the two considered approaches generate the same B-spline basis in $S(\hat{\mathcal{U}}_m, \hat{\mathcal{M}}, \hat{\Delta})$. However, proceeding in the former way entails that the initial EC-system on $[x_\ell, x_{\ell+1})$ must be stored in order to be able to perform any subsequent operation in $S(\hat{\mathcal{U}}_m, \hat{\mathcal{M}}, \hat{\Delta})$.

4. Order elevation

Let $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ and $S(\mathcal{U}_{m+r}, \widetilde{\mathcal{M}}, \Delta)$, $r \geq 1$, be two spline spaces good for design such that $S(\mathcal{U}_m, \mathcal{M}, \Delta) \subset S(\mathcal{U}_{m+r}, \widetilde{\mathcal{M}}, \Delta)$. By means of the transition functions we will show how to express the B-spline basis $\{N_{i,m}\}_i$ of $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ as a linear combination of the elements of the B-spline basis $\{N_{i,m+r}\}_i$ of $S(\mathcal{U}_{m+r}, \widetilde{\mathcal{M}}, \Delta)$, where $\widetilde{\mathcal{M}}$ is obtained from \mathcal{M} by simply increasing by r the multiplicity of each break-point. In particular, the basic steps of the order elevation algorithm are:

1. subdivide the spline into pieces expressed in terms of the Bernstein basis;
2. order elevate each spline piece;
3. remove the redundant knots.

To accomplish step 1, it is sufficient to recall that a representation in the Bernstein basis can be derived by performing repeated knot insertions, as illustrated in the previous section. Thus, in the following, we will focus on the second step of the algorithm, which requires

providing a relation between the Bernstein bases of order m and order $m+r$. We will focus on the two cases $r = 1, 2$, since elevation to any arbitrary order can be performed by combining several steps of 1- or 2-order elevations. In particular, a direct formula to accomplish elevation by two orders at once can be useful in the context of piecewise Chebyshevian spline spaces. In fact, on the one hand there exist Bernstein bases of EC-spaces of dimension m that cannot be expressed via convex combinations of elements of Bernstein bases of EC-spaces of dimension $m+1$ [25]. On the other hand, 2-order elevation turns out to be useful when working with spaces containing trigonometric and/or hyperbolic functions, because such functions usually appear in couple among the generators of the space. The following result holds in the case $r = 1$.

Proposition 2. *Let \mathcal{U}_{n+1} and \mathcal{U}_{n+2} be respectively $(n+1)$ - and $(n+2)$ -dimensional good for design spaces on $[a, b] \subset \mathbb{R}$, with $\mathcal{U}_{n+1} \subset \mathcal{U}_{n+2}$. Let $\{B_{i,n}\}_{i=0,\dots,n}$ and $\{B_{i,n+1}\}_{i=0,\dots,n+1}$ be the Bernstein bases for \mathcal{U}_{n+1} and \mathcal{U}_{n+2} . Then there exist coefficients $\{\gamma_0, \dots, \gamma_{n+1}\}$, $0 \leq \gamma_i \leq 1$, $i = 0, \dots, n+1$, such that*

$$B_{i,n}(x) = \gamma_i B_{i,n+1}(x) + (1 - \gamma_{i+1}) B_{i+1,n+1}(x), \quad i = 0, \dots, n, \quad x \in [a, b]. \quad (15)$$

In particular,

$$\gamma_i = \begin{cases} 1, & i = 0, \\ \frac{D^i g_i(a)}{D^i \tilde{g}_i(a)}, & i = 1, \dots, n, \\ 0, & i = n+1, \end{cases} \quad (16)$$

where g_i , $i = 0, \dots, n$, and \tilde{g}_i , $i = 0, \dots, n+1$, are the transition functions in \mathcal{U}_{n+1} and \mathcal{U}_{n+2} .

Proof. In an EC-space, each Bernstein basis functions $B_{i,n}$, $i = 0, \dots, n$, has at the endpoints of $[a, b]$ zeroes of respective multiplicity i and $n-i$. Hence, the relation between the bases of orders $n+1$ and $n+2$ has the form

$$B_{i,n}(x) = \gamma_i B_{i,n+1}(x) + \delta_{i+1} B_{i+1,n+1}(x), \quad x \in [a, b].$$

In fact, if $B_{i,n}$ was a combination of additional basis functions, it could not have at a and b the right number of zeroes. By construction we know that at the point a , $B_{0,n}$ and $B_{0,n+1}$ are equal to 1, while $B_{1,n+1}$ is equal to 0. There follows that $\gamma_0 = 1$. A similar reasoning at b leads to $\delta_{n+1} = 1$. In addition, we know that the Bernstein bases are positive for all $x \in (a, b)$ and that $D^i B_{i,n}(a) > 0$ for $i = 1, \dots, n$, therefore $\gamma_i \geq 0$. A similar reasoning at b leads to $\delta_{i+1} \geq 0$. Since

$$\begin{aligned}
1 &\equiv \sum_{i=0}^n B_{i,n}(x) \\
&= \sum_{i=0}^n (\gamma_i B_{i,n+1}(x) + \delta_{i+1} B_{i+1,n+1}(x)) \\
&= \sum_{i=0}^{n+1} B_{i,n+1}(x),
\end{aligned}$$

for all $x \in [a, b]$, we have $\gamma_0 = \delta_{n+1} = 1$ and $\gamma_i + \delta_i = 1$ for $i = 1, \dots, n$, proving that

$$\gamma_i = \begin{cases} 1, & i = 0, \\ \frac{D^i B_{i,n}(a)}{D^i B_{i,n+1}(a)}, & i = 1, \dots, n, \\ 0, & i = n+1, \end{cases}$$

from which (16) immediately follows. \square

An immediate consequence of relation (15) is the following.

Corollary 2. *Under the same assumptions of Proposition 2, let*

$$p(x) = \sum_{i=0}^n c_i B_{i,n}(x) = \sum_{i=0}^{n+1} \tilde{c}_i B_{i,n+1}(x), \quad x \in [a, b].$$

Then

$$\tilde{c}_i = \begin{cases} c_0, & i = 0, \\ \gamma_i c_i + (1 - \gamma_i) c_{i-1}, & i = 1, \dots, n, \\ c_n, & i = n+1, \end{cases} \quad (17)$$

with $\gamma_i, i = 1, \dots, n$, given by (16).

Elevation by two orders can be performed as stated below.

Proposition 3. *Let \mathcal{U}_{n+1} and \mathcal{U}_{n+3} be respectively $(n+1)$ - and $(n+3)$ -dimensional goof for design spaces on $[a, b] \subset \mathbb{R}$, with $\mathcal{U}_{n+1} \subset \mathcal{U}_{n+3}$. Let $\{B_{i,n}\}_{i=0,\dots,n}$ and $\{B_{i,n+2}\}_{i=0,\dots,n+2}$ be the Bernstein bases for \mathcal{U}_{n+1} and \mathcal{U}_{n+3} . Then there exist coefficients $\{\gamma_i\}_{i=0,\dots,n+2}$ and $\{\delta_i\}_{i=1,\dots,n+2}$, such that*

$$\begin{aligned}
B_{i,n}(x) &= \gamma_i B_{i,n+2}(x) + \delta_{i+1} B_{i+1,n+2}(x) + \\
&\quad (1 - \gamma_{i+2} - \delta_{i+2}) B_{i+2,n+2}(x), \quad (18) \\
i &= 0, \dots, n, \quad x \in [a, b].
\end{aligned}$$

In particular,

$$\gamma_i = \begin{cases} 1, & i = 0, \\ \frac{D^i g_i(a)}{D^i \tilde{g}_i(a)}, & i = 1, \dots, n, \\ 0, & i = n+1, n+2, \end{cases} \quad (19a)$$

$$\delta_{i+1} = \begin{cases} 1 - \gamma_1, & i = 0, \\ \gamma_i - \gamma_{i+1} + \frac{D^{i+1} g_i(a) - \gamma_i D^{i+1} \tilde{g}_i(a)}{D^{i+1} \tilde{g}_{i+1}(a)}, & i = 1, \dots, n, \\ 0, & i = n+1, \end{cases} \quad (19b)$$

where $g_i, i = 0, \dots, n$, and $\tilde{g}_i, i = 0, \dots, n+2$ are the transition functions in \mathcal{U}_{n+1} and \mathcal{U}_{n+3} .

Proof. In an EC-space, each Bernstein basis functions $B_{i,n}, i = 0, \dots, n$, has at the endpoints of $[a, b]$ zeroes of respective multiplicity i and $n - i$. Hence, the relation between the bases of order $n+1$ and $n+3$ has the form

$$B_{i,n}(x) = \gamma_i B_{i,n+2}(x) + \delta_{i+1} B_{i+1,n+2}(x) + \epsilon_{i+2} B_{i+2,n+2}(x), \quad x \in [a, b].$$

In fact, if $B_{i,n}$ was a combination of additional basis functions, it could not have a and b as zeros of the right multiplicity. By construction we know that, at the point a , $B_{0,n}$ and $B_{0,n+2}$ are equal to 1, while $B_{1,n+2}$ and $B_{2,n+2}$ are equal to 0. There follows that $\gamma_0 = 1$. A similar reasoning at b yields $\epsilon_{n+2} = 1$. In addition, we know that the bases are positive for all $x \in (a, b)$, that $D^i B_{i,n}(a) > 0$ for $i = 1, \dots, n$ and $D^i B_{i,n+2}(a) > 0$ for $i = 1, \dots, n+2$, therefore $\gamma_i \geq 0$. Since

$$\begin{aligned}
1 &\equiv \sum_{i=0}^n B_{i,n}(x) \\
&= \sum_{i=0}^n (\gamma_i B_{i,n+2}(x) + \delta_{i+1} B_{i+1,n+2}(x) + \epsilon_{i+2} B_{i+2,n+2}(x)) \\
&= \sum_{i=0}^{n+2} B_{i,n+2}(x),
\end{aligned}$$

for all $x \in [a, b]$, we have $\gamma_0 = \epsilon_{n+2} = 1$, $\gamma_1 + \delta_1 = 1$, $\delta_{n+1} + \epsilon_{n+1} = 1$, and $\gamma_i + \delta_i + \epsilon_i = 1$, for $i = 2, \dots, n$, proving that

$$\gamma_i = \begin{cases} 1, & i = 0, \\ \frac{D^i B_{i,n}(a)}{D^i B_{i,n+2}(a)}, & i = 1, \dots, n, \\ 0, & i = n+1, n+2, \end{cases}$$

$$\delta_{i+1} = \begin{cases} 1 - \gamma_1, & i = 0, \\ \frac{D^{i+1} B_{i,n}(a) - \gamma_i D^{i+1} B_{i,n+2}(a)}{D^{i+1} B_{i+1,n+2}(a)}, & i = 1, \dots, n, \\ 0, & i = n+1, \end{cases}$$

from which (19a) and (19b) follow. \square

An immediate consequence of relation (18) is the following result.

Corollary 3. *Under the same assumptions of Proposition 3, let*

$$p(x) = \sum_{i=0}^n c_i B_{i,n}(x) = \sum_{i=0}^{n+2} \tilde{c}_i B_{i,n+2}(x), \quad x \in [a, b].$$

Then

$$\tilde{c}_i = \begin{cases} c_0, & i = 0, \\ \gamma_1 c_1 + (1 - \gamma_1) c_0, & i = 1, \\ \gamma_i c_i + \delta_i c_{i-1} + (1 - \gamma_i - \delta_i) c_{i-2}, & i = 2, \dots, n, \\ \delta_{n+1} c_n + (1 - \delta_{n+1}) c_{n-1}, & i = n + 1, \\ c_n, & i = n + 2, \end{cases} \quad (21)$$

with $\gamma_i, i = 1, \dots, n$, as in (19a) and $\delta_i, i = 2, \dots, n + 1$, as in (19b).

5. Computational methods

The constructive approach proposed in this paper straightforwardly translates into efficient numerical procedures. These include the algorithms presented below for the evaluation of splines with nonuniform and possibly multiple knots (and of their derivatives and integrals) and for the computation of the coefficients of the representations associated with a refined knot partition or with a raised order.

Algorithm 1 (Construction of the transition functions). Let $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ be a piecewise Chebyshevian spline space and Δ^* an extended partition. To determine the coefficients of the transition functions $f_i, i = 2, \dots, m + K$, w.r.t. the EC-systems $\{u_{j,1}, \dots, u_{j,m}\}, j = 0, \dots, q$:

1. solve the linear system (8) for each $f_i, i = 2, \dots, m + K$;
2. assemble the matrix $B = (b_{i,j})$ of dimensions $(m + K) \times m(m - 1)$, where the i th row contains the solution of the i th linear system, i.e., the coefficients that define the transition function f_i w.r.t. to the EC-systems $\{u_{j,1}, \dots, u_{j,m}\}, j = p_i, \dots, p_{i+m-1} - 1$.

Remark 5. Note that the first row of the above matrix B is not defined nor used in the following algorithms, since it corresponds to the transition function $f_i \equiv 1$.

Algorithm 2 (Evaluation of the transition functions). To evaluate at $\bar{x} \in [t_\ell, t_{\ell+1}] = [x_{p_\ell}, x_{p_{\ell+1}})$ the (nontrivial) transition functions $f_{\ell-m+2}, \dots, f_\ell$, it is necessary to evaluate their pieces $f_{\ell-m+2, p_\ell}, \dots, f_{\ell, p_\ell}$:

1. **for** $i = \ell - m + 2, \dots, \ell$
 - i. $r \leftarrow m(p_\ell - p_i)$;
 - ii. $f_i(\bar{x}) \leftarrow \sum_{h=1}^m b_{i, r+h} u_{p_\ell, h}(\bar{x})$ (see equation (7)), where the $b_{i, r+h}$'s are the entries of the matrix B constructed by Algorithm 1.

Algorithm 3 (Evaluation of a piecewise Chebyshevian spline). Given $\bar{x} \in [a, b]$, to evaluate a spline of the form (1) at \bar{x} :

1. determine ℓ such that $\bar{x} \in [t_\ell, t_{\ell+1})$;
2. use Algorithm 2 to evaluate the transition functions $f_{\ell-m+2}, \dots, f_\ell$ at \bar{x} ;
3. $N_{\ell-m+1, m}(\bar{x}) \leftarrow 1 - f_{\ell-m+2}(\bar{x})$;
 $N_{i, m}(\bar{x}) \leftarrow f_i(\bar{x}) - f_{i+1}(\bar{x})$ for $i = \ell - m + 2, \dots, \ell - 1$;
 $N_{\ell, m}(\bar{x}) \leftarrow f_\ell(\bar{x})$.
4. perform the linear combination in (1).

Remark 6 (Derivatives and integral of a piecewise Chebyshevian spline). By linearity, evaluating derivatives and integrals amounts to differentiating and integrating the transition functions and, ultimately, the generators $u_{i,j}$ of the EC-spaces. This computation follows the same outline of Algorithms 2 and 3.

Algorithm 4 (Knot insertion). Let $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ be a given piecewise Chebyshevian spline space, Δ^* an extended partition, and s a spline function of the form (1). Given $\hat{t} \in [a, b]$, to determine the coefficients $\hat{c}_i, i = 1, \dots, m + K + 1$, obtained by inserting \hat{t} in Δ^* once:

1. determine ℓ such that $\hat{t} \in [t_\ell, t_{\ell+1})$;
2. compute the coefficients $\alpha_i, i = \ell - m + 2, \dots, \ell - r + 1$ through (11);
3. use (14) to compute the coefficients $\hat{c}_i, i = 1, \dots, m + K + 1$.

Algorithm 5 (Order elevation). Let $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ be a given piecewise Chebyshevian spline space, Δ^* an extended partition, and s a spline function of the form (1). To represent s in the space $S(\mathcal{U}_{m+q}, \widetilde{\mathcal{M}}, \Delta)$, whose order is elevated by $r = 1, 2$:

1. repeatedly apply Algorithm 4 to the space $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ until all the knots have multiplicity $n = m - 1$;
2. on each nontrivial knot interval $[t_\ell, t_{\ell+1}]$, determine the transition functions $g_i, i = 1, \dots, m$, of $\mathcal{U}_{\ell, m}$, and evaluate at t_ℓ the derivative of order i if $r = 1$, and also of order $i + 1$ if $r = 2$;
3. on each nontrivial knot interval $[t_\ell, t_{\ell+1}]$, determine the transition functions $\tilde{g}_i, i = 1, \dots, m + r$, of $\mathcal{U}_{\ell, m+r}$, and evaluate at t_ℓ the derivative of order i if $r = 1$, and also of order $i + 1$ if $r = 2$;

4. if $r = 1$, determine the coefficients $\gamma_i, i = 1, \dots, n$, through (16), or, if $r = 2$, determine the coefficients $\gamma_i, i = 1, \dots, n$, through (19a) and the coefficients $\delta_i, i = 1, \dots, n + 1$, through (19b);
5. determine the coefficients $\tilde{c}_i, i = 0, \dots, m + r$, either through (17) if $r = 1$ or through (21) if $r = 2$;
6. apply a knot removal algorithm to the spline space $S(\mathcal{U}_{m+r}, \tilde{\mathcal{M}}, \Delta)$, in which all the knots have multiplicity m , until multiplicity $\mu_i + r$ is obtained, where μ_i is the initial multiplicity of each knot in $S(\mathcal{U}_m, \mathcal{M}, \Delta)$.

6. Application examples

This section is a collection of several examples aimed at illustrating different aspects of the proposed approach. Example 3 presents in detail all the necessary steps to construct the transition functions (and thus the B-spline basis) of a given piecewise Chebyshevian spline space. Example 4 shows that, in simpler cases, the transition functions and the B-spline basis can be determined by symbolic computation, thus obtaining explicit formulas. The knot insertion and order elevation procedures presented in Sections 3 and 4 are illustrated through Examples 5, 6, 7, with a view to applications in Isogeometric Analysis and Geometric Modeling. Finally, Example 8 aims at illustrating the advantageous use of piecewise Chebyshevian spline surfaces.

All spaces considered in the following examples are good for design. This can be verified by existing theoretical results (see [7]) for spaces of dimension up to 4, or by means of computational procedures for spaces of higher dimension, using the numerical algorithm in [10] or its direct generalization to the spline setting.

Example 3 (Construction of a B-spline function). In this example we will show the construction of a single B-spline function defined on a given sequence of EC-spaces. In particular, we consider the three spaces

$$\begin{aligned} \mathcal{U}_{0,3} &= \text{span}\{1, t, t^2\}, & t &= t(x) := x - x_0, \\ & & x &\in [x_0, x_1], \\ \mathcal{U}_{1,3} &= \text{span}\{1, \cos(\theta t), \sin(\theta t)\}, & t &= t(x) := x - x_1, \\ & & x &\in [x_1, x_2], \theta(x_2 - x_1) < \pi, \\ \mathcal{U}_{2,3} &= \text{span}\{1, \cosh(\phi t), \sinh(\phi t)\}, & t &= t(x) := x - x_2, \\ & & x &\in [x_2, x_3], \end{aligned}$$

and, for the ease of presentation, we assume all break-points to have multiplicity 1. Denoted as usual $\Delta^* = \{t_j\}$ the extended partition, we derive the expression of the B-spline function $N_{3,3} = f_3 - f_4$ having support

$[t_3, t_6] = [x_0, x_3]$. Expanding $f_j, j = 3, 4$, in the EC-system $\{u_{\ell,1}, u_{\ell,2}, u_{\ell,3}\}, \ell = j, j + 1$, we obtain

$$f_j(x) = \begin{cases} 0, & x \leq t_j, \\ \sum_{k=1}^3 b_{j,k} u_{j-3,k}(t(x)), & t_j \leq x < t_{j+1}, \\ \sum_{k=1}^3 b_{j,k+3} u_{j-2,k}(t(x)), & t_{j+1} \leq x < t_{j+2}, \\ 1, & x \geq t_{j+2}. \end{cases}$$

Next we compute the coefficients $b_{j,k}$ of the transition function $f_j, j = 3, 4$, by solving the two 6×6 linear systems of the form (8) that arise from the endpoint and continuity conditions

$$\begin{aligned} f_{j,1}(x_{j-3}) &= 0, & Df_{j,1}(x_{j-3}) &= 0, \\ f_{j,1}(x_{j-2}) &= f_{j,2}(x_{j-2}), & Df_{j,1}(x_{j-2}) &= Df_{j,2}(x_{j-2}), \\ f_{j,2}(x_{j-1}) &= 1, & Df_{j,2}(x_{j-1}) &= 0. \end{aligned} \quad (22)$$

Solving (22) for f_3 we obtain

$$\begin{aligned} b_{3,1} &= b_{3,2} = 0, \\ b_{3,3} &= \frac{1}{h_0^2 + 2\frac{h_0}{\theta} \tan\left(\frac{\theta}{2}h_1\right)}, \\ b_{3,4} &= 1 + \frac{1}{\cos(\theta h_1) - \frac{\theta}{2}h_0 \sin(\theta h_1) - 1}, \\ b_{3,5} &= (1 - b_{3,4}) \cos(\theta h_1), \\ b_{3,6} &= \frac{1}{\frac{\theta}{2}h_0 + \tan\left(\frac{\theta}{2}h_1\right)}, \end{aligned}$$

while, for f_4 ,

$$\begin{aligned} b_{4,1} &= -b_{4,2} = \frac{1}{1 - \cos(\theta h_1) + \frac{\theta}{\phi} \sin(\theta h_1) \tanh\left(\frac{\phi}{2}h_2\right)}, \\ b_{4,3} &= 0, \\ b_{4,4} &= 1 + \frac{1}{\cosh(\phi h_2) + \frac{\phi}{\theta} \tan\left(\frac{\theta}{2}h_1\right) \sinh(\phi h_2) - 1}, \\ b_{4,5} &= (1 - b_{4,4}) \cosh(\phi h_2), \\ b_{4,6} &= \frac{1}{\frac{\phi}{\theta} \tan\left(\frac{\theta}{2}h_1\right) + \tanh\left(\frac{\phi}{2}h_2\right)}, \end{aligned}$$

where $h_i = t_{i+1} - t_i$. An illustration of the transition functions and of the related B-spline bases is given in Figure 2 for $\theta = 2, \phi = 4$ (right) and $\theta = 10, \phi = 15$ (left). We remark that, for these values of the parameters, the method in [26] can be used to verify that the considered piecewise Chebyshevian spline space is good for design.

Example 4 (Explicit formulae for B-spline bases in the case $m = 4$). In Section 5 we have provided compu-

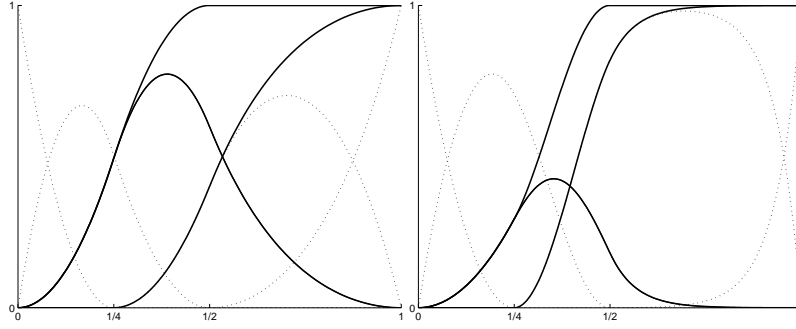


Figure 2: B-spline basis and transition functions f_3 and f_4 involved in the construction of $N_{3,3}$ according to (4) in the spline space $S(\mathcal{U}_3, \mathcal{M}, \Delta)$ with $\mathcal{U}_3 := \{\mathcal{U}_{0,3}, \mathcal{U}_{1,3}, \mathcal{U}_{2,3}\}$ defined in Example 3 and extended partition $\Delta^* = \{0, 0, 0, \frac{1}{4}, \frac{1}{2}, 1, 1, 1\}$. On the left $\theta = 2$ and $\phi = 4$, on the right $\theta = 10$ and $\phi = 15$.

tational algorithms that apply to any piecewise Chebyshevian spline space, regardless of the type or dimension of the EC-systems. However, if we consider spaces with a relatively simple structure (that is, low dimension and basic expressions for the generators), the theoretical framework also allows for explicitly determining the B-spline basis by symbolic computation. As an example, we can obtain the B-spline basis functions corresponding to nonuniform partitions and break-points of multiplicities 1 for some spaces widely used in the context of geometric design. To the best of our knowledge these expressions have never appeared before.

Proposition 4. *Let us consider a piecewise Cheby-*

shevian spline space having 4-dimensional sections of type $\mathcal{U}_{i,4} = \text{span}\{1, t, u_i(t), v_i(t)\}$, with $t = \frac{x-x_i}{x_{i+1}-x_i}$ and $x \in [x_i, x_{i+1}]$, and where all $u_i(t)$ and $v_i(t)$ belong to one of the following types:

- A) $u_i(t) = \cos(\theta_i t)$ and $v_i(t) = \sin(\theta_i t)$, with $\theta_i \in (0, \pi)$;
- B) $u_i(t) = \cosh(\phi_i t)$ and $v_i(t) = \sinh(\phi_i t)$, $\theta_i \in \mathbb{R}$;
- C) $u_i(t) = \frac{(1-t)^3}{1+(v_i-3)(1-t)}$ and $v_i(t) = \frac{t^3}{1+(v_i-3)(1-t)}$, with $v_i \geq 3$.

For arbitrarily spaced knots $\{t_i\}_i$, each B-spline basis function $N_{i,4}$ has support $[t_i, t_{i+4}]$ and is of the form

$$N_{i,4}(x) = \begin{cases} \frac{1}{H_i} \frac{h_i^2 F_i(\tau_i(x))}{h_i + h_{i+1}}, & t_i \leq x < t_{i+1}, \\ -\left(\frac{1}{H_i} + \frac{1}{H_{i+1}}\right) \frac{h_{i+1}^2 F_{i+1}(\tau_{i+1}(x))}{h_{i+1} + h_{i+2}} - \frac{1}{H_i} \left(G_{i+1}(\tau_{i+1}(x)) - \frac{h_{i+1}^2 F_{i+1}(1 - \tau_{i+1}(x))}{h_i + h_{i+1}} \right), & t_{i+1} \leq x < t_{i+2}, \\ 1 - \left(\frac{1}{H_i} + \frac{1}{H_{i+1}}\right) \frac{h_{i+2}^2 F_{i+2}(1 - \tau_{i+2}(x))}{h_{i+1} + h_{i+2}} + \frac{1}{H_{i+1}} \left(G_{i+2}(\tau_{i+2}(x)) + \frac{h_{i+2}^2 F_{i+2}(\tau_{i+2}(x))}{h_{i+2} + h_{i+3}} \right), & t_{i+2} \leq x < t_{i+3}, \\ \frac{1}{H_{i+1}} \frac{h_{i+3}^2 F_{i+3}(1 - \tau_{i+3}(x))}{h_{i+2} + h_{i+3}}, & t_{i+3} \leq x < t_{i+4}, \\ 0, & \text{otherwise,} \end{cases}$$

where $h_i := t_{i+1} - t_i$, $\tau_i(x) := \frac{x-t_i}{h_i}$, and, depending on the case, the other quantities are respectively defined as follows:

A)

$$\begin{aligned} F_i(t) &= \theta t - \sin(\theta t), \\ G_i(t) &= (\theta - \sin \theta)(h_i - h_{i-1}) - \theta(1 - \cos \theta)h_i t, \\ H_i &= (\theta - \sin \theta)(h_i + h_{i+1} + h_{i+2}) + \\ &\quad (3 \sin \theta - \theta(2 + \cos \theta))h_{i+1}; \end{aligned}$$

B)

$$\begin{aligned} F_i(t) &= \theta t - \sinh(\theta t), \\ G_i(t) &= (\theta - \sinh \theta)(h_i - h_{i-1}) - \theta(1 - \cosh \theta)h_i t, \\ H_i &= (\theta - \sinh \theta)(h_i + h_{i+1} + h_{i+2}) + \\ &\quad (3 \sinh \theta - \theta(2 + \cosh \theta))h_{i+1}; \end{aligned}$$

C)

$$\begin{aligned} F_i(t) &= \frac{t^3}{1 + (h_i - 3)(1 - t)t}, \\ G_i(t) &= (h_i - h_{i-1}) - h_i h_i t, \\ H_i &= (h_i + h_{i+1} + h_{i+2}) + (h_i - 3)h_{i+1}. \end{aligned}$$

Remark 7. For break-points with multiplicities $\mu_i > 1$, the B-spline basis can be obtained as the limit of the expressions above, when the knot configuration approaches one with a proper number of coincident knots. Alternatively, the expression of the B-spline basis functions can also be obtained by performing the symbolic computation for scratch, starting from a partition with coincident knots.

Example 5 (Tools for Isogeometric Analysis). This example aims at illustrating the benefits of the proposed approach in the context of Isogeometric Analysis.

A first advantage is related to the ease of computation of integrals and derivatives. More precisely, once the derivatives and antiderivatives of the EC-systems $\{u_{j,1}, \dots, u_{j,m}\}$, $j = 0, \dots, q$, are known, this computation follows along the same lines of Algorithms 2 and 3 (see also Remark 1). In this respect, it shall also be considered that piecewise Chebyshevian spline spaces commonly used in geometric design are based on trigonometric, polynomial, or simple rational functions and thus the derivatives and primitive functions of the EC-systems have a simple form.

Moreover, by analogy with the polynomial spline case, the tools of knot insertion and order elevation provided in Sections 3 and 4 can be combined to perform the so-called k -refinement [27, Section 2.6], which results in lower number of basis functions and higher continuity compared to the classical h - p -refinement. This is illustrated in Figure 3, which is analogous to Figure 10 in [27], with the only difference that piecewise Chebyshevian splines are used instead of polynomial ones.

Example 6 (2-order elevation). As discussed in Section 4, when the target space contains an additional couple of trigonometric or hyperbolic functions, it can be reached elevating by two orders at once. Figure 4 shows two examples of 2-order elevations for both Bézier and piecewise Chebyshevian B-spline curves built on spaces of mixed type.

Example 7 (Modeling with piecewise Chebyshevian splines). This example illustrates two essential tools for geometric modeling and design. The first is the refinement of a parametric curve, which is shown in Figures 5(a) and 5(b). The second is the procedure of conversion into standard representation (clamped knot partition) of a closed piecewise Chebyshevian spline curve with a periodic nonuniform knot partition. It has been obtained exploiting the knot insertion formulae (11) and (13) respectively at the left and right endpoint of the knot partition. The standard representation of Figure 5(b) is shown in Figure 5(c).

Example 8 (Piecewise Chebyshevian splines surfaces). Any standard algorithm for polynomial splines can be applied by substituting the polynomial B-spline basis by its piecewise Chebyshevian counterpart. This example aims at illustrating the advantageous use of piecewise Chebyshevian splines in constructing tensor product surfaces. Figure 6(a) shows a profile curve composed by 8 section curves corresponding to a uniform periodic knot partition, with an alternation of section spaces of the form $\text{span}\{1, \cos(\theta t), \sin(\theta t)\}$, $\theta = \pi/2$ and $\text{span}\{1, t, t^2\}$. As a consequence, the resulting spline space has section spaces of different type. In particular, the “smoothed corners” of the section curve in Figure 6(a) are exact circular arcs, whereas the straight segments are described in terms of quadratic polynomials. The extended partition $\{0, 0, 0, 1, 1, 1\}$ is associated to the other direction of the parametric domain with section space $\text{span}\{1, \cos(\phi t), \sin(\phi t)\}$, $\phi = \pi/2$. Figures 6(b) and 6(c) show two views of the resulting tensor product piecewise Chebyshevian spline surface with the control net. Note that, the resulting surface can also be obtained by means of NURBS. However the NURBS representation requires 16×3 control points and yields a G^1 surface, whereas the representation in terms of piecewise Chebyshevian splines (besides not involving rational functions) is C^1 and only needs 10×3 control points.

7. Extension to more general spline spaces

In this section we will briefly discuss how the transitions functions can be constructed and effectively exploited in more general contexts. This means that we will still be able to benefit from the computational advantages as well as from the tools of knot insertion and order elevation presented in the previous sections.

By no means this intends to be an exhaustive discussion. We will just content ourselves to illustrating the main necessary modifications in the construction of

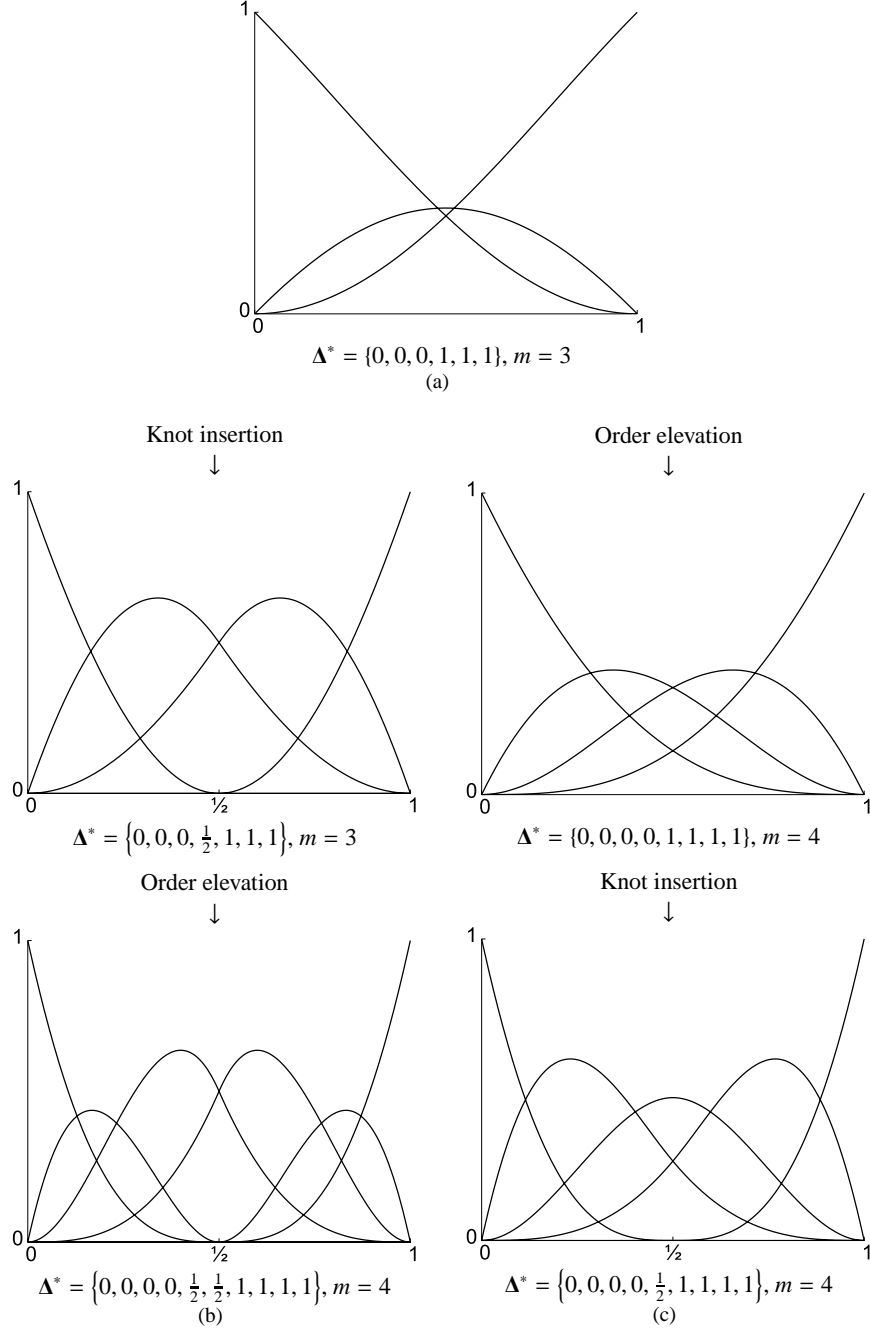


Figure 3: *k*-refinement takes advantage of the fact that knot insertion and order elevation do not commute. (a) Base case of one element in the EC-space span $\{1, \cos(\theta t), \sin(\theta t)\}$ with $\theta = 2$ and the specified extended partition Δ^* . The EC-space is to be order-elevated to span $\{1, t, \cos(\theta t), \sin(\theta t)\}$. (b) *h*-*p*-refinement: knot insertion followed by order elevation results in six blending functions that are C^1 at the break-point $\frac{1}{2}$. (c) *k*-refinement: order elevation followed by knot insertion results in five blending functions that are C^2 at the break-point $\frac{1}{2}$.

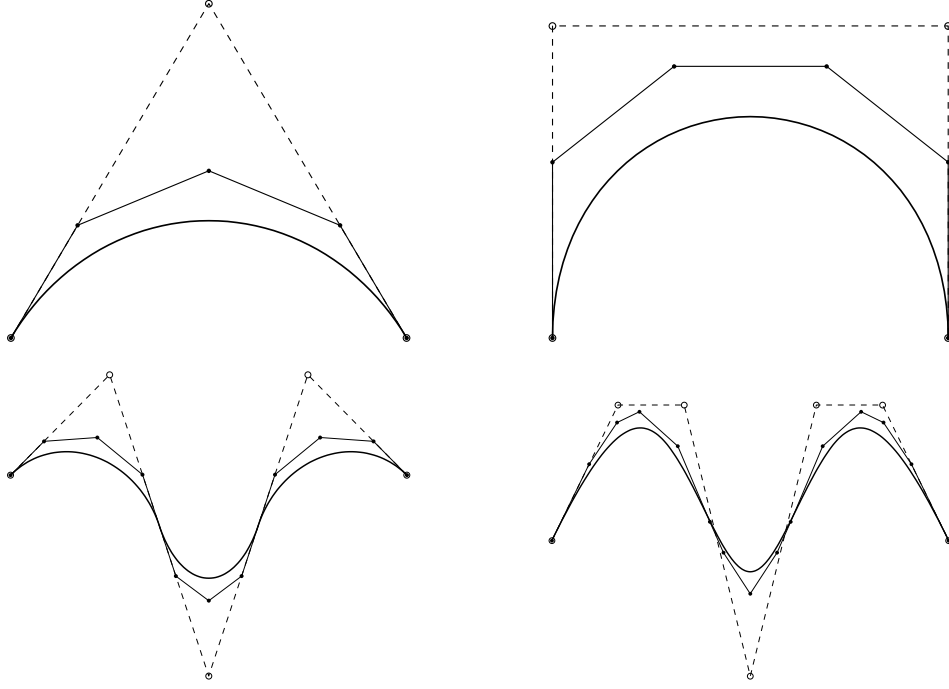


Figure 4: Elevation by two orders on Bézier (top) and piecewise Chebyshevian B-spline (bottom) curves for different spaces: from $\text{span}\{1, \cos(\theta_i t), \sin(\theta_i t)\}$ to $\text{span}\{1, \cos(\theta_i t), \sin(\theta_i t), \cosh(\phi_i t), \sinh(\phi_i t)\}$ with $\theta_i = 2$, $\phi_i = 1$, $\forall i$ (left), from $\text{span}\{1, t, \cos(\theta_i t), \sin(\theta_i t)\}$ to $\text{span}\{1, t, \cos(\theta_i t), \sin(\theta_i t), t \cos(\theta_i t), t \sin(\theta_i t)\}$ with $\theta_i = 2$, $\forall i$ (right). For the B-spline curves in the bottom left and right figures the extended partitions are $\Delta^* = \{0, 0, 0, \frac{1}{3}, \frac{2}{3}, 1, 1, 1\}$ and $\Delta^* = \{0, 0, 0, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1, 1, 1\}$ respectively.

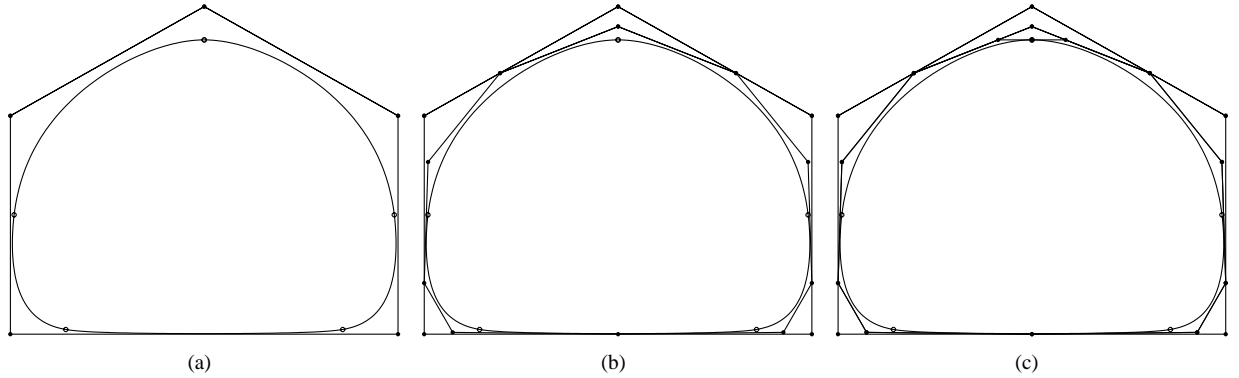


Figure 5: Example of a closed spline curve with periodic nonuniform knot partition. All the underlying EC-spaces are $\text{span}\left\{1, t, \frac{(1-t)^3}{1+(v_i-3)(1-t)t}, \frac{t^3}{1+(v_i-3)(1-t)t}\right\}$, where the tension parameter v_i is 8 for the space in $[0.45, 0.55]$ and 4 for all others. Left: curve and control polygon corresponding to the extended partition $\Delta^* = \{-0.55, -0.45, -0.35, 0, 0.35, 0.45, 0.55, 0.65, 1, 1.35, 1.45, 1.55\}$; center: one step of refinement, obtained by inserting the midpoint of each knot interval; right: standard representation.

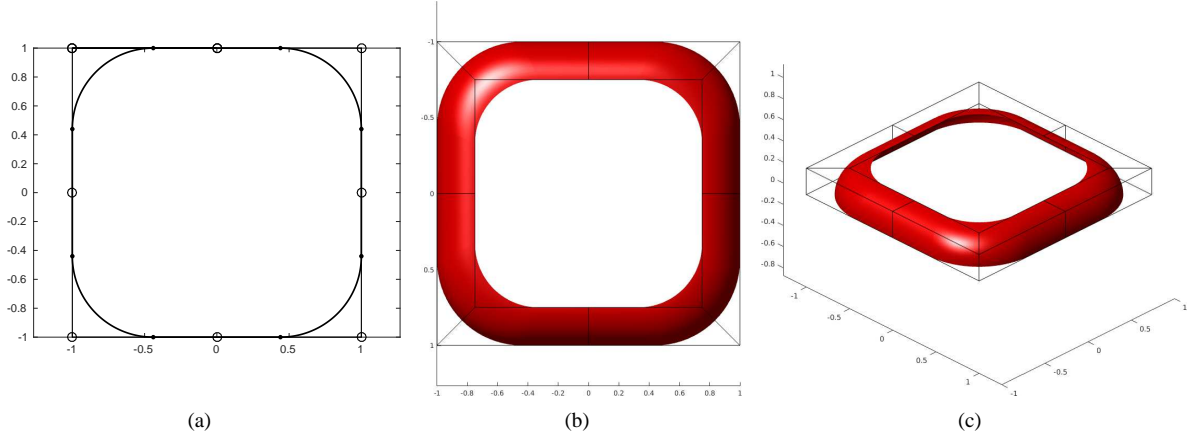


Figure 6: Tensor product surface (and its profile curve) obtained by means of piecewise Chebyshevian splines described in Example 8.

the transition functions and some numerical examples based on a generalized version of the proposed computational procedures.

We would like to emphasize that all the examples presented in this section are obtained using the transition functions. This is worthy of interest, since we are not aware of any alternative general procedure for the practical computation of splines in the spaces that we will consider.

7.1. Geometrically Continuous piecewise Chebyshevian splines

So far we have focused on splines where adjacent pieces are connected with parametric continuity, which means that their derivatives at break-points must agree up to proper order of continuity. Parametrically continuous splines are an important subclass of the wider set of *geometrically continuous* splines (studied e.g. in [6] and [28]) and this section aims to explore how the tools developed so far can be extended to this more general context.

In addition to the setting and notation introduced in Section 2.2, we shall now associate with the elements of Δ a sequence $\mathbf{M} := (M_1, \dots, M_q)$ of *connection matrices*, where M_i , $i = 1, \dots, q$ is lower triangular of order $m - \mu_i$, has positive diagonal entries and first row and column equal to $(1, 0, \dots, 0)$. Hence a space of *geometrically continuous piecewise Chebyshevian splines*, which we indicate by $S(\mathcal{U}_m, \mathcal{M}, \Delta, \mathbf{M})$, is formally the same as in Definition 3, but replacing condition ii) with the following

ii)

$$\begin{aligned} M_i \left(D^0 s_{i-1}(x_i), D^1 s_{i-1}(x_i), \dots, D^{m-\mu_{i-1}} s_{i-1}(x_i) \right)^T &= \\ &= \left(D^0 s_i(x_i), D^1 s_i(x_i), \dots, D^{m-\mu_i} s_i(x_i) \right)^T, \\ i &= 1, \dots, q. \end{aligned}$$

Clearly, when all matrices M_i are the identity, the considered splines are parametrically continuous. In addition, note that the requirement that the first row and column of each matrix M_i be equal to the vector $(1, 0, \dots, 0)$ guarantees the continuity of the splines.

As a consequence of the above point ii), every transition function f_i , $i = 1, \dots, m + K$ will be required to satisfy the continuity conditions

$$\begin{aligned} M_j \left(D^0 f_{i,j-1}(x_j), D^1 f_{i,j-1}(x_j), \dots, D^{k_j} f_{i,j-1}(x_j) \right)^T &= \\ &= \left(D^0 f_{i,j}(x_j), D^1 f_{i,j}(x_j), \dots, D^{k_j} f_{i,j}(x_j) \right)^T, \\ j &= p_i + 1, \dots, p_{i+m-1} - 1, \end{aligned}$$

in place of the parametric continuity conditions (6). Taking into account this modification, it is possible to generalize all the tools developed in the preceding sections in order to work within the framework of geometric continuity.

Remark 8. When performing knot-insertion, as described in Section 3 we shall proceed as follows. If a new knot is inserted so as to increase the multiplicity of an existing knot, the related connection matrix must be updated by removing its last row and column. When a new knot is inserted at a location that does not correspond to any already existing knot, then the corresponding connection matrix must be the identity matrix.

The following is an example of geometrically continuous curves modelled by means of transition functions.

Example 9 (Geometric continuity). Let us consider the EC-spaces $\mathcal{T}_4 = \text{span}\{1, t, \cos(t), \sin(t)\}$ and $\mathcal{P}_4 = \text{span}\{1, t, t^2, t^3\}$ and the piecewise Chebyshevian spline space $S(\mathcal{U}_4, \mathcal{M}, \Delta, \mathbf{M})$ defined on $[0, 3]$ with $\mathcal{U}_4 := \{\mathcal{T}_4, \mathcal{P}_4, \mathcal{P}_4, \mathcal{T}_4\}$, break-points $\Delta = \{1, \frac{3}{2}, 2\}$, multiplicities $\mathcal{M} = (2, 1, 2)$ and connection matrices $\mathbf{M} = (M_1, M_2, M_3)$ having the form:

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix},$$

with $\beta \in \mathbb{R}$. Figures 7 (a), (b) and (c) show the B-spline basis functions corresponding to $\beta = -7, 0, 14$. Figures 7 (d), (e) and (f) depict the corresponding spline curves defined by a “C”-shaped control polygon. The multiplicities of break-points and the structure of the connection matrices make so that the curves are symmetric, G^1 at the break-points 1 and 2 and G^2 at $\frac{3}{2}$. The superimposed curvature comb emphasizes the G^1 and G^2 smoothness properties at the joint between two spline segments. In the case $\beta = 0$ the curve is C^2 at $3/2$, whereas it is G^1 at the other knots. It is also interesting to notice the “tension” effect obtained for increasing values of the parameter β which affects the curve shape in the intervals $[1, \frac{3}{2}]$ and $[\frac{3}{2}, 2]$.

7.2. Break-points with multiplicity equal to zero

In a polynomial spline space, whenever $\mu_i = 0$, then there is no break-point at x_i because the polynomial part of $s \in S(\mathcal{P}_m, \mathcal{M}, \Delta)$ on $[x_{i-1}, x_i]$ determines s uniquely on $[x_i, x_{i+1}]$. In the case of piecewise Chebyshevian splines the situation is not the same, because different EC-spaces may exist on $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$. This entails that there can be a break-point x_i , which separates the intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$, with no corresponding knot t_j in the extended partition Δ^* . In this case, we say that the break-point x_i has *zero multiplicity*. The splines in the corresponding space will be C^{m-1} at x_i . Piecewise Chebyshevian splines with knots of zero multiplicity have been deeply studied (see e.g. [7] and references therein).

As concerns the construction of the transition functions, this can be accomplished following the outline of Section 2.2. In particular, it will be sufficient to take $\mu_i = 0$ at the location of break-points with zero multiplicity in (6).

Example 10 (Zero multiplicity). The spline space considered in this example is defined on the interval $[0, 4]$, has break-points $\Delta = \{1, 2, 3\}$ with multiplicities $\mathcal{M} = (1, 0, 1)$ and is spanned on every interval $[x_i, x_{i+1}]$, $i = 0, \dots, 3$, by the space of rational functions

$\text{span}\left\{1, t, \frac{(1-t)^3}{1+(v_i-3)(1-t)}, \frac{t^3}{1+(v_i-3)(1-t)}\right\}$, where $t = \frac{x-x_i}{x_{i+1}-x_i}$, $v_0 = v_3 = 4$ and $v_1 = v_2 = 6$ (note that v_i , which also appears at point C) in Proposition 4, shall be greater or equal than 3 [24]). It can be shown by standard arguments that the considered spline space has dimension 6. Figure 8(a) shows the B-spline basis, while Figure 8(b) depicts the spline curve generated with 6 control points on a square. The curvature comb emphasizes that the curve is C^2 at the break-points 1 and 3, and C^3 at the break-point 2, which has zero multiplicity.

It is also possible to consider break points of zero multiplicity in spaces of geometrically continuous splines. In this case, also at such break points the continuity conditions have the form 7.1, depending on a connection matrix. Space that are good for design and where all break points have zero multiplicities provide interesting shape effects, as recently investigated in [10].

7.3. Multi-order piecewise Chebyshevian splines

In the polynomial case, spline-like functions comprised of polynomial segments of various degrees, called *multi-degree splines*, were firstly proposed in [29]. Recently, the B-spline basis for these spaces was computed by means of an integral recurrence formula [30] and used to define *changeable degree splines* [31]. B-spline bases for spline spaces built upon local ECT-systems of different types and orders, called *multi-order splines*, were studied in [23] and [18]. In the case of multi-order splines it is more natural to select the order of continuity k_i at break-points, rather than the multiplicity of knots, under the condition that $0 \leq k_i < \min(m_{i-1}, m_i)$ for $i = 1, \dots, q$, being m_i the order of the EC-space assigned to $[x_i, x_{i+1}]$. As discussed in [18], for constructing the B-spline basis it is convenient to consider two different extended partitions $\Delta_t^* = \{t_i\}$ and $\Delta_s^* = \{s_i\}$, such that each break-point x_i is repeated precisely $(m_i - k_i - 1)$ times in Δ_t^* and $(m_{i-1} - k_i - 1)$ times in Δ_s^* , and where x_0 is repeated m_0 times in Δ_t^* and x_{q+1} , m_q times in Δ_s^* . We will therefore take two sequences

$$\{t_1, \dots, t_K\} \equiv \underbrace{\{x_0, \dots, x_0\}}_{m_0 \text{ times}}, \underbrace{\{x_1, \dots, x_1\}}_{m_1 - k_1 - 1 \text{ times}}, \dots, \underbrace{\{x_q, \dots, x_q\}}_{m_q - k_q - 1 \text{ times}},$$

and

$$\{s_1, \dots, s_K\} \equiv \underbrace{\{x_1, \dots, x_1\}}_{m_0 - k_1 - 1 \text{ times}}, \dots, \underbrace{\{x_q, \dots, x_q\}}_{m_{q-1} - k_q - 1 \text{ times}}, \underbrace{\{x_{q+1}, \dots, x_{q+1}\}}_{m_q \text{ times}}.$$

In this way the dimension of a multi-order generalized spline space is simply the cardinality of Δ_t^* or of Δ_s^* .

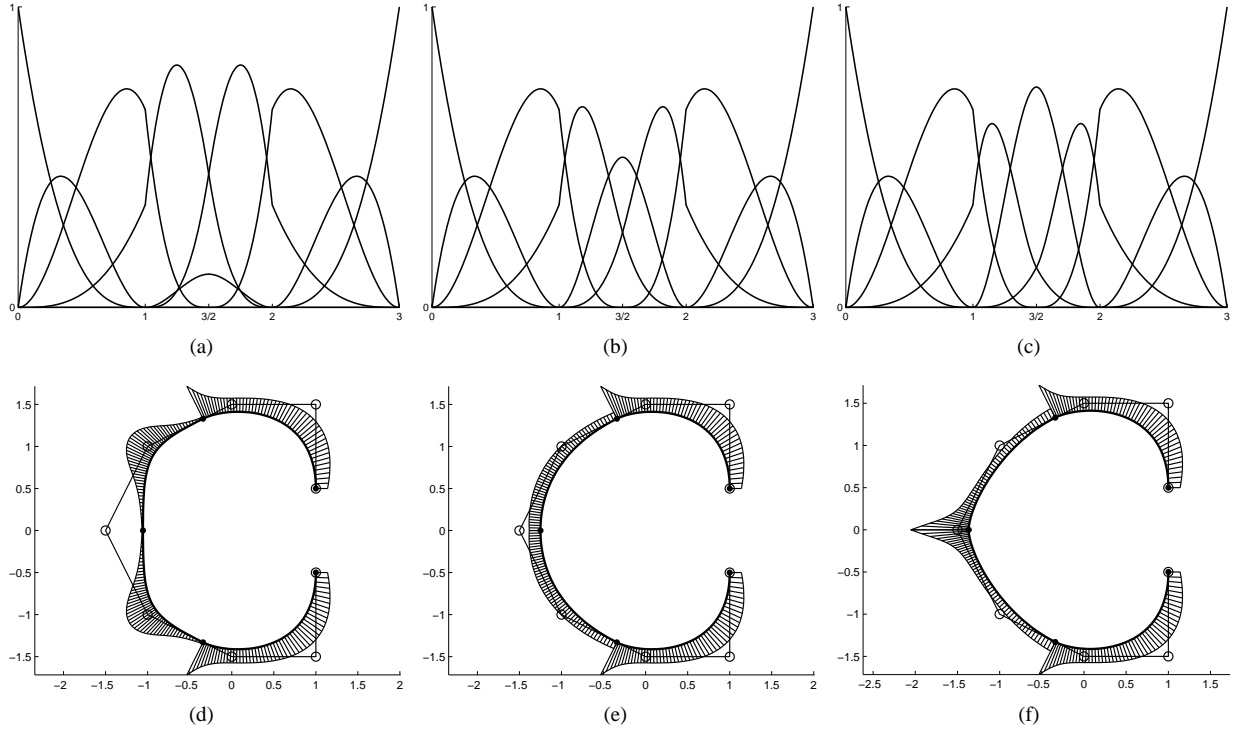


Figure 7: Geometrically continuous B-spline functions and open spline curves for the spline spaces in Example 9. Figures (a), (b) and (c) show the B-spline functions for $\beta = -7, 0, 14$ respectively. Figures (d), (e) and (f) show the corresponding curves and curvature comb.

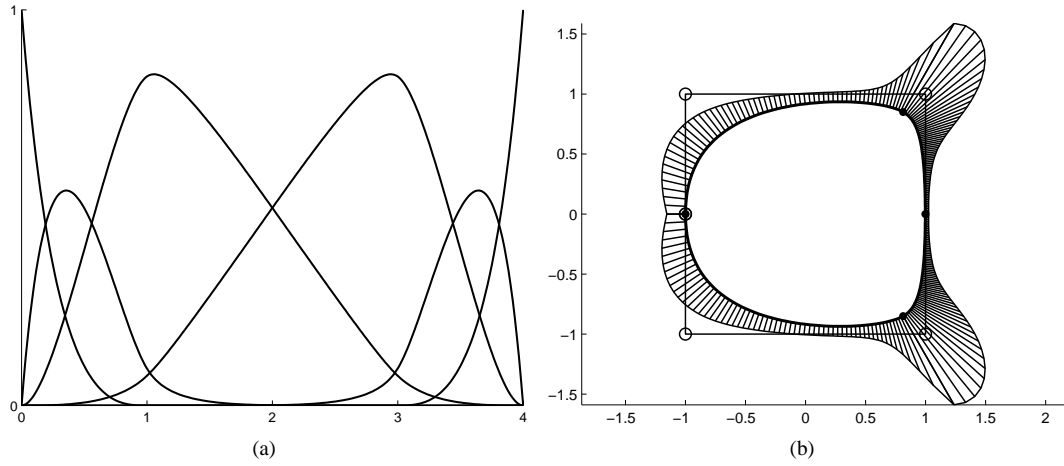


Figure 8: B-spline functions in the piecewise Chebyshevian spline spaces of Example 10 and corresponding closed spline curve. In Figure (a) the B-spline functions with break-points $\Delta = \{1, 2, 3\}$ and multiplicities $M = (1, 0, 1)$ are shown. In Figure (b) curve, control polygon starting from $(-1, 0)$, and curvature comb are shown.

Moreover each knot of Δ_t^* and Δ_s^* is respectively the starting point and endpoint of a B-spline basis function, namely the interval $[t_i, s_i]$ is the support of the i th B-spline N_i , $i = 1, \dots, K$. In this setting, each transition function f_i , $i = 2, \dots, K$, is nontrivial in the interval (t_i, s_{i-1}) and can be determined from an adapted version of the conditions (6).

The following is an interesting example of a multi-order spline.

Example 11 (Multi-order spline). This example aims at illustrating two important facts: on the one hand, we wish to emphasize that multi-order spline spaces can be approached by means of transition functions, similarly as the other spaces considered so far. On the other hand, we will demonstrate how multi-order splines can be used to represent special curves, by taking on each segment the EC-space of the minimum possible order. To this aim, we have designed a curve composed of five distinct pieces, including a cardioid segment, a straight line segment, two circular arcs and lastly a cubic segment. The spline space is defined on a uniform knot partition of $[0, 5]$ with the following sequence of local spaces

$$\begin{aligned} &\text{span}\{1, t\}, \quad \text{span}\{1, \cos(\theta t), \sin(\theta t)\}, \\ &\text{span}\{1, \cos(\theta t), \sin(\theta t)\}, \quad \text{span}\{1, t, t^2, t^3\}, \\ &\text{span}\{1, \cos(\phi t), \sin(\phi t), \cos(2\phi t), \sin(2\phi t)\}, \end{aligned}$$

with $\theta = \pi/2$ and $\phi = 2/3\pi$. The result is a closed, non-periodic curve. C^1 continuity was required at break-points. Figures 9(a) and 9(b) show respectively the B-spline basis and the designed curve together with the cardioid and the circle (in dashed line) whose segments are exactly reproduced.

7.4. Piecewise Quasi Chebyshevian spline spaces

QEC-spaces are a superset of EC-spaces and differ from the latter in that they do not permit Taylor interpolation. They are formally defined as follows [32].

Definition 7 (Quasi Extended Chebyshev space). Let $I \subset \mathbb{R}$ be a closed bounded interval. An m -dimensional space $\mathcal{U}_m \subset C^{m-2}(I)$, $m \geq 2$, is a *Quasi Extended Chebyshev space* (QEC-space, for short) on I if any Hermite interpolation problem in m data in I , with at least two distinct points, has a unique solution in \mathcal{U}_m . Equivalently, for $m > 2$, \mathcal{U}_m is a QEC-space if any nonzero element of \mathcal{U}_m with at least two distinct zeros vanishes at most $m-1$ times in I counting multiplicities.

The definition of a piecewise quasi Chebyshevian spline space is formally the same as Definition 3, with

the sole modification that $\mathcal{U}_{i,m}$, $i = 0, \dots, q$, will be QEC-spaces. In the recent work [22] necessary and sufficient conditions to establish when a piecewise quasi Chebyshevian spline space is *good for design* were given. For such a space, a *quasi B-spline basis* is a sequence $\{N_{i,m}\}_{i=1, \dots, m+K}$ of elements of $S(\mathcal{U}_m, \mathcal{M}, \Delta)$ which meets the requirements in Definition 5 where property iii) is replaced by:

- iii) *endpoint property*: for each $i \in \{1, \dots, m+K\}$, $N_{i,m}$ vanishes $m - \mu_i^R$ times at t_i (exactly if $\mu_i^R > 1$) and $m - \mu_{i+m}^L$ times at t_{i+m} (exactly if $\mu_{i+m}^L > 1$).

When a space has the quasi B-spline basis, we can define the transition functions as in (2). However, in order to generalize the algorithms proposed in the previous sections, it is important to bear in mind the following important difference compared to the piecewise Chebyshevian setting. For any transition function f_i , let \bar{k}_i^R be the first integer such that $D_+^r f_i(t_i) = 0$, $r = 0, \dots, \bar{k}_i^R$, $D_+^{\bar{k}_i^R+1} f_i(t_i) > 0$. If $\mathcal{U}_{i,m}$ is a QEC-space, but not an EC-space, then $\bar{k}_i^R \geq k_i^R$. Analogously, defined \bar{k}_{i+m-1}^L the last integer such that $D_-^r f_i(t_{i+m-1}) = 0$, $r = 1, \dots, \bar{k}_{i+m-1}^L$, $(-1)^{\bar{k}_{i+m-1}^L} D_-^{\bar{k}_{i+m-1}^L+1} f_i(t_{i+m-1}) > 0$, then there holds $\bar{k}_{i+m-1}^L \geq k_{i+m-1}^L$. This peculiarity of the transition functions affects for example the computation of the knot-insertion coefficients $\alpha_{i,m}$, because it requires us to estimate the minimum order of differentiation \bar{k}_i^R or \bar{k}_{i+m-1}^L to be considered in (11) or (13).

As an example of a QEC-space, let us consider the $(n+1)$ -dimensional space of functions

$$\mathcal{U}_{n+1} := \text{span}\{1, t, \dots, t^{n-2}, (1-t)^{n_1}, t^{n_2}\}, \quad t \in [0, 1],$$

for $n > 1$, $n_1, n_2 \geq n-2$. When n_1, n_2 are integers, we obtain the so-called *variable-degree polynomial spaces* [33]. These were firstly introduced for the purpose of *shape preserving* spline interpolation with $n = 3$ and $n_1 = n_2$, later on without the restriction $n_1 = n_2$. The presence of the shape parameters n_1, n_2 also makes these spaces interesting for geometric design purposes. Further properties of the latter spaces with any real number as exponents were investigated in [34].

Example 12 (Piecewise quasi Chebyshevian splines). Let us consider a spline space on the interval $[0, 6]$ with uniformly-spaced break points x_i , $i = 1, \dots, 5$, and where any two consecutive sections are of the form

$$\begin{aligned} &\text{span}\{1, \cos(\theta t), \sin(\theta t), t \cos(\theta t), t \sin(\theta t)\}, \\ &\text{span}\{1, t, t^2, (1-t)^n, t^n\}, \end{aligned}$$

with $t \in \frac{x-x_i}{x_{i+1}-x_i}$, $x \in [x_i, x_{i+1}]$, $i = 0, \dots, 5$. The former space is an EC-space good for design for $\theta \leq 2\pi$, while

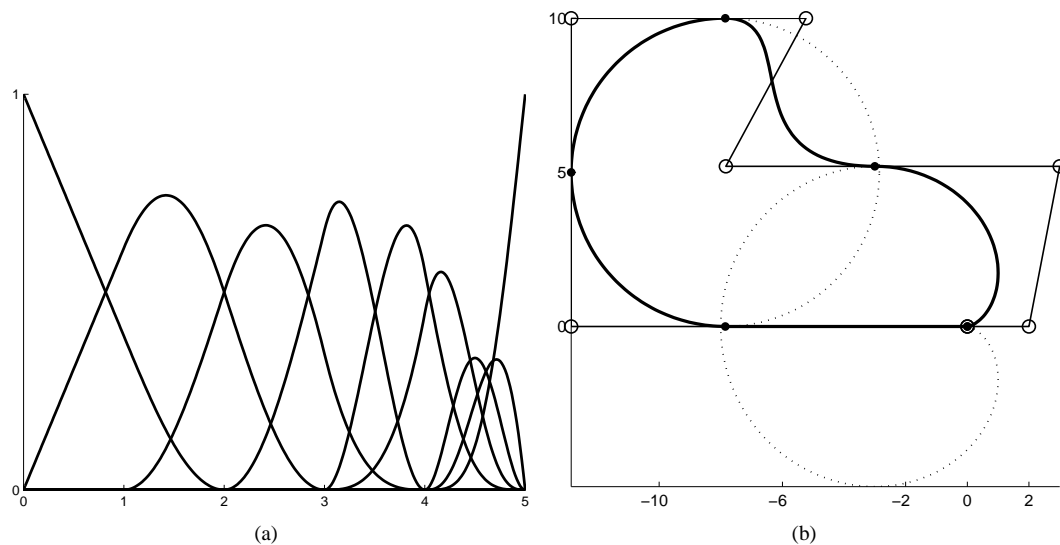


Figure 9: B-spline functions and closed spline curve in the multi-order generalized spline space given in Example 11. In Figure (a) the B-spline functions of the multi-order generalized spline space and in Figure (b) the designed curve and its control polygon are shown.

the latter is a QEC-space of so-called *variable-degree splines* [34]. C^3 continuity is required at break-points. We have verified experimentally that such a spline space is good for design. Figure 10 shows the B-spline basis, computed by means of transition functions, and closed periodic curves corresponding to $n = 5, 10, 50, \theta = 1$. For increasing values of n an interesting tension effect is obtained.

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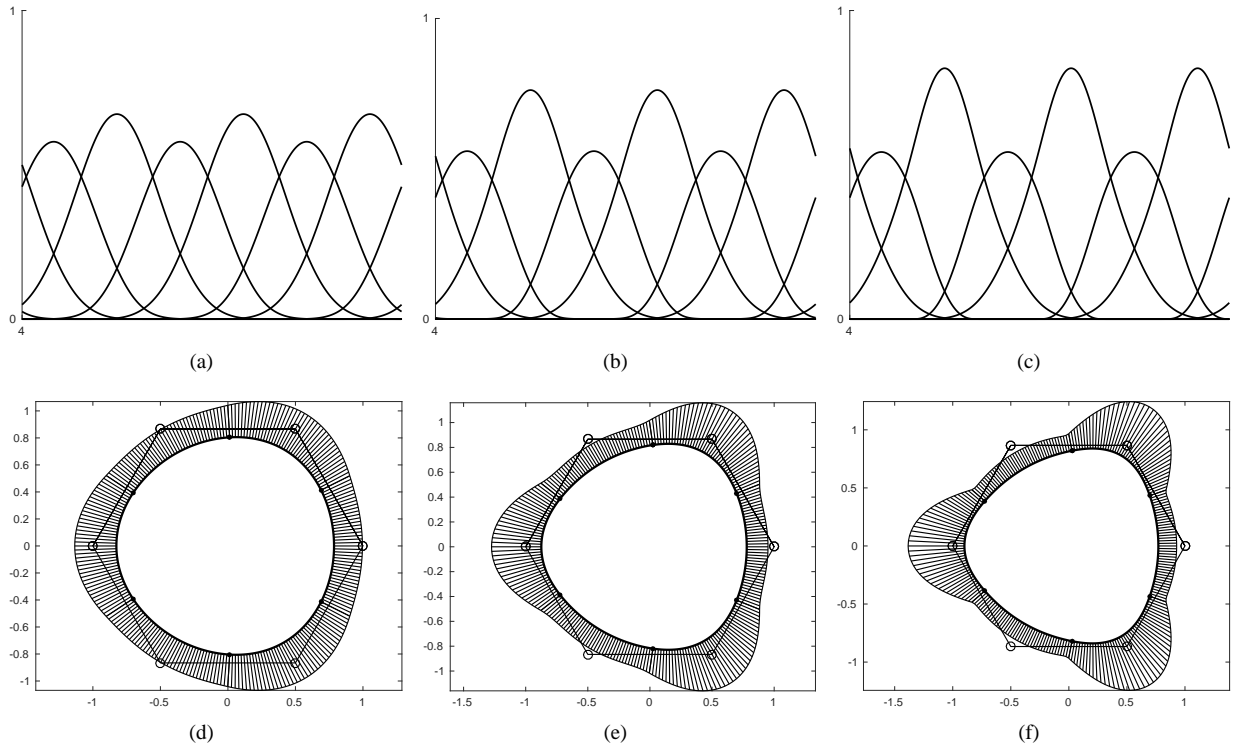


Figure 10: B-spline functions and closed spline curves in the piecewise quasi Chebyshevian spline spaces given in Example 12. Figures (a), (b) and (c) show the B-spline functions of the spaces corresponding respectively to $n = 5, 10, 50$, $\theta = 1$. Figures (d), (e) and (f) show the corresponding closed curves with tension effect.

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